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Calculating the particle-field correlation in a flowing plasma

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Abstract. This paper presents a quasi-linear derivation of the correlation between the fluctuations in the magnetic field and the phase space density, $\langle \delta \boldsymbol{B} \delta f \rangle$, applicable to an infinitely conducting, moving plasma; that is, the derivation includes the effect of the electric field, $\boldsymbol{E} = -\boldsymbol{V}/c \times \boldsymbol{B}$, where $\boldsymbol{V}$ is the plasma velocity.

1 Introduction

The relativistic Vlasov equation, written in three-vector notation, equals

$$\frac{\partial f}{\partial t} + v^{i} \frac{\partial f}{\partial x^{i}} + F^{i} \frac{\partial f}{\partial p^{i}} = 0, \tag{1}$$

where the phase space coordinates, x^i and p^i, as well as the particle velocity, v^i, and the force, F^i, are all three-vectors, and t is the coordinate time. Plasma turbulence consists of highly dynamic and variable fluctuations and other departures from the large-scale leading order behavior in the magnetic and electric fields; this in turn causes fluctuations and variability in the particle momentum and distribution function. In addition, exact knowledge of the physical quantities, F^i and f, does not exist. Consequently, statistical methods must be used in the study of solar wind turbulence. With this motivation, let

$$\begin{aligned} F^{i} &= \langle F^{i} \rangle + \delta F^{i}, \\ f &= \langle f \rangle + \delta f, \end{aligned} \tag{2}$$

where the angle brackets, $\langle \cdots \rangle$, represents an ensemble average and the δ terms represent the fluctuating component. By definition, the ensemble average of the fluctuating component equals zero identically; that is, $\langle \delta F^i \rangle = \langle \delta f \rangle \equiv 0$.

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The method of characteristics returns the formal solution for the fluctuating component of the phase space density,

$$\delta f(\boldsymbol{x}, \boldsymbol{p}, t) = \delta f(0) - \int_0^t \delta F^{i}(t')\, \partial_{p^{i}} \Big\langle f(t') \Big\rangle\, dt', \tag{3}$$

where

$$\begin{aligned} x^{i}(t') &= x^{i}(0) + \int_0^{t'} v^{i}(t'')\, dt'' \\ p^{i}(t') &= p^{i}(0) + \int_0^{t'} \Big\langle F^{i}(\boldsymbol{x}'', \boldsymbol{p}'', t'') \Big\rangle\, dt''. \end{aligned} \tag{4}$$

For brevity, we suppress the coordinate dependence of δF^i and $\langle f \rangle$ in Eq. 3; it should be understood that quantities in the integrand depend on the position $\boldsymbol{x}(t')$, momentum $\boldsymbol{p}(t')$, and coordinate time, t'.

In terms of the Lorentz force, the expression for δf equals

$$\begin{aligned} \delta f(t) = \delta f(0) &- e \int_0^t \delta E^{i}(t')\, \partial_{p^{i}} \Big\langle f(t') \Big\rangle\, dt' \\ &- \frac{e}{c} \int_0^t \varepsilon^{i}{}_{jk}(t')\, v^{j}(t')\, \delta B^{k}(t')\, \partial_{p^{i}} \Big\langle f(t') \Big\rangle\, dt', \end{aligned} \tag{5}$$

where $\varepsilon^{i}{}_{jk}$ is the third-rank Levi-Civita tensor.

2 The Particle-Field Correlation

Bieber (1987) focuses on the correlation between fluctuations of the magnetic field and fluctuations of the particle distribution function, $\langle \delta B \delta f \rangle$, a quantity that plays a role in quasi-linear theory. This measurable quantity affords an opportunity to test quasi-linear theory at a fundamental level; moreover, it can provide unique information on the detailed nature of interplanetary magnetic turbulence.

To obtain $\langle \delta B \delta f \rangle$, multiply Eq. 5 by $\delta B^K(\boldsymbol{x}, \boldsymbol{p}, t)$ and ensemble average. This returns

$$\begin{aligned} \Big\langle \delta B^{K} \delta f \Big\rangle(t) = &-e \int_0^t \Big\langle \delta B^{K}(t) \delta E^{i}(t') \Big\rangle\, \partial_{p^{i}} \Big\langle f(t') \Big\rangle dt' \\ &- \frac{e}{c} \int_0^t \varepsilon^{i}{}_{jk}(t')\, v^{j}(t') \Big\langle \delta B^{K}(t) \delta B^{k}(t') \Big\rangle\, \partial_{p^{i}} \Big\langle f(t') \Big\rangle dt' \end{aligned} \tag{6}$$

where

$$\left\langle \delta B^K(t)\,\delta f(0)\right\rangle = \delta f(0)\left\langle \delta B^K(t)\right\rangle = 0. \tag{7}$$

Note that δB^K is evaluated at time t, not t'; therefore, δB^K may be placed inside the integrand.

Because experiments readily measure the two-point correlation tensor, various turbulence models describe turbulent fluctuations in terms of a two-point correlation tensor. In spatially homogeneous and time stationary turbulence, bulk translation of the measurement apparatus does not affect the statistical properties of the turbulence; that is, the statistical properties do not depend on the observation point, they only depend on the spatial and temporal separation between two measurement points, $\boldsymbol{X}$ and T, respectively. Matthaeus and Goldstein (1982b) have shown that the solar wind satisfies the conditions of weak stationarity if the effects of solar rotation are included; weak stationarity exists if the first and second moments of the probability distribution are themselves time stationary. By definition, the magnetic correlation tensor and the mixed electric-magnetic correlation tensor equals

$$\begin{aligned} R^{Kk}_{BB}(\boldsymbol{X},T) &= \left\langle \delta B^K(\boldsymbol{x},t)\,\delta B^k(\boldsymbol{x}+\boldsymbol{X},t+T)\right\rangle,\\ R^{K\imath}_{BE}(\boldsymbol{X},T) &= \left\langle \delta B^K(\boldsymbol{x},t)\,\delta E^\imath(\boldsymbol{x}+\boldsymbol{X},t+T)\right\rangle. \end{aligned} \tag{8}$$

An additional change of variable, $t \to t-T$, results in $R^{Kk}_{BB}(T) = R^{kK}_{BB}(-T)$.

To recast Eq. 6 in terms of the two-point correlation tensor, use the change of variable $t' = t - T$. With this change of variable, Eq. 6 equals

$$\begin{aligned} \left\langle \delta B^K \delta f\right\rangle(t) &= -e\int_0^t R^{\imath K}_{EB}(T)\,\partial_{p^\imath}\left\langle f(t')\right\rangle dT\\ &\quad -\frac{e}{c}\int_0^t \varepsilon^\imath{}_{\jmath k}(t')\,v^\jmath(t')\,R^{kK}_{BB}(T)\,\partial_{p^\imath}\left\langle f(t')\right\rangle dT. \end{aligned} \tag{9}$$

Note that the change of variable also affects Eq. 4; with this change of variable $x^\imath(t')$ and $p^\imath(t')$ transform to

$$\begin{aligned} x^\imath(t') &= x^\imath(0) + \int_0^t v^\imath(t'')\,dt'' + \int_t^{t-T} v^\imath(t'')\,dt''\\ &= x^\imath(t) + \Delta^\imath_x\\ p^\imath(t') &= p^\imath(0) + \int_0^t \left\langle F^\imath(t'')\right\rangle dt'' + \int_t^{t-T}\left\langle F^\imath(t'')\right\rangle dt''\\ &= p^\imath(t) + \Delta^\imath_p. \end{aligned} \tag{10}$$

When the correlation tensors, $R^{kK}_{BB}(T)$ and $R^{\imath K}_{EB}(T)$, satisfy the conditions of a Lanczos-type function (Matthaeus and Goldstein, 1982a),

$$\begin{aligned} R^{kK}_{BB}(T) &= \begin{cases} R^{kK}_{BB}(T) & |T| < T_c\\ \sim 0 & |T| > T_c \end{cases}\\ R^{\imath K}_{EB}(T) &= \begin{cases} R^{\imath K}_{EB}(T) & |T| < T_c^\star\\ \sim 0 & |T| > T_c^\star \end{cases} \end{aligned} \tag{11}$$

then $\langle \delta B^K \delta f\rangle(t)$ contains two important time domains. For $t < \max(T_c, T_c^\star)$, $\langle \delta B^K \delta f\rangle$ is given by Eq. 9. On the other hand, for $t \geq \max(T_c, T_c^\star)$, then $T_{CM} \equiv \max(T_c, T_c^\star)$ can replace t as the upper limit of the integrals in Eq. 9; that is,

$$\begin{aligned} \left\langle \delta B^K \delta f\right\rangle(t) &= -e\int_0^{T_{CM}} R^{\imath K}_{EB}(T)\,\partial_{p^\imath}\left\langle f(t')\right\rangle dT\\ &\quad -\frac{e}{c}\int_0^{T_{CM}} \varepsilon^\imath{}_{\jmath k}(t')\,v^\jmath(t')\,R^{kK}_{BB}(T)\,\partial_{p^\imath}\left\langle f(t')\right\rangle dT \end{aligned} \tag{12}$$

Equation 12 assumes a simple form in an infinitely conducting, moving plasma, such as the solar wind, where $\boldsymbol{E} = -\boldsymbol{V}_{SW}/c \times \boldsymbol{B}$. Consequently,

$$\delta\boldsymbol{E} = -\frac{V_{SW}B}{c}\left(\hat{\boldsymbol{V}}_{SW}\times\frac{\delta\boldsymbol{B}}{B} + \frac{\delta\boldsymbol{V}_{SW}}{V_{SW}}\times\hat{\boldsymbol{B}}\right). \tag{13}$$

Observations indicate that δV_{SW} is of the order of the Alfvén speed, ~ 30 km/s, and that the average solar wind speed is ~ 400 km/s, while $\delta B/B \sim 1$; therefore, the second term can be neglected. Under these conditions, Eq. 12 equals

$$\begin{aligned} \left\langle \delta B^K \delta f\right\rangle(t) &= -\frac{e}{c}\int_0^{T_{CM}} \varepsilon^\imath{}_{\jmath k}(t') \quad \times\\ &\quad \left[v^\jmath(t') - V^\jmath_{SW}(t')\right] R^{kK}_{BB}(T)\,\partial_{p^\imath}\left\langle f(t')\right\rangle dT \end{aligned} \tag{14}$$

The remainder of this discussion will focus on Eq. 14.

More specifically, consider a moving plasma with a homogeneous magnetic field. Orient the coordinate system such that $\langle \boldsymbol{B}\rangle = B_0\hat{\boldsymbol{z}}$ and $\boldsymbol{V}_{SW} = V_{SW}\sin\psi\,\hat{\boldsymbol{x}} + V_{SW}\cos\psi\,\hat{\boldsymbol{z}}$, where ψ is the garden-hose field angle. It is customary to express the momentum gradient, $\partial_{p^\imath}\langle f\rangle$, in spherical coordinates. Orient the spherical coordinate system such that $\boldsymbol{p} = p\,\boldsymbol{\epsilon}^1$. Then $\boldsymbol{V}_{SW}$, expressed in terms of a coordinate basis, instead of the usual orthonormal basis, equals

$$\begin{aligned} V^1_{SW} &= V_{SW}\Big(\sin\psi\sin\theta\cos\phi + \cos\psi\cos\theta\Big),\\ V^2_{SW} &= \frac{V_{SW}}{p}\Big(\sin\psi\cos\theta\cos\phi - \cos\psi\sin\theta\Big),\\ V^3_{SW} &= -\frac{V_{SW}}{p}\frac{\sin\psi}{\sin\theta}\sin\phi, \end{aligned} \tag{15}$$

where θ is the particle pitch angle, and ϕ is the particle gyrophase. The non-zero elements of the Levi-Civita tensor equal

$$\begin{aligned} \varepsilon^1{}_{23} &= r^2\sin\theta = -\varepsilon^1{}_{32},\\ \varepsilon^2{}_{31} &= \sin\theta = -\varepsilon^2{}_{13},\\ \varepsilon^3{}_{12} &= \frac{1}{\sin\theta} = -\varepsilon^3{}_{21}. \end{aligned} \tag{16}$$

For a homogeneous magnetic field, it can be shown that

$$p = p_I\left[1 + \beta\mathcal{H}^{(1)}_p\right] \tag{17}$$

to first-order in $\beta = \beta_{SW}\sin\psi$, where

$$\mathcal{H}^{(1)}_p = \frac{\varepsilon_I}{cp_I}\sin\theta_I\Big[\cos\big(G_\Omega - \phi_I\big) - \cos\phi_I\Big]. \tag{18}$$

By definition, $G_\Omega \equiv \Omega t - k_x(x - x_I)$, where $k_x \equiv \beta\Omega/c$ and

$$\Omega = \frac{ecB}{\varepsilon_I}\Big(1 + \beta\frac{cp_I}{\varepsilon_I}\sin\theta_I\cos\phi_I\Big) \tag{19}$$

to first-order in β. In the above equations ε is the total particle energy, including the rest energy, and the subscript I indicates the initial conditions of the particle motion.

Express the particle density as a sum of anisotropic contributions, f_0, and isotropic contributions, $\xi^{(A)}$,

$$\Big\langle f(\boldsymbol{x},\boldsymbol{p},t)\Big\rangle = f_0(\boldsymbol{x},p,t)\Big[1+\sum_A \xi^{(A)}(\boldsymbol{x},\boldsymbol{p},t)\Big] \tag{20}$$

where the isotropic particle density equals $f_0 = kp^{-\gamma}$. In general, $k = k(\boldsymbol{x},t)$ and $\gamma = \gamma(\boldsymbol{x},p,t)$. The spectral index, γ, is momentum dependent because the primary cosmic ray flux can not be described by a single power law. But, according to Eq. 17, the change in momentum equals 0.1–10%, depending on the particle's initial momentum; therefore, for a given particle momentum, the momentum dependence of the spectral index can be neglected. Furthermore, the time variation of k and γ, as well as $\xi^{(A)}$, slowly occurs over time periods from a year to a solar cycle; because these time periods are much, much longer than T_{CM}, the time dependence of k, γ, and $\xi^{(A)}$ can also be neglected. Bieber and Pomerantz (1983) have calculated that the diurnal anisotropy, $\xi^{(1)}$, has an amplitude of 0.714%, the semi-diurnal anisotropy, $\xi^{(2)}$, has an amplitude of 0.051%, and the tridiurnal anisotropy, $\xi^{(3)}$, has an amplitude of 0.018%. The anisotropic terms, $\xi^{(A)}$ are frequently expressed in terms of the real spherical harmonics.

From Eq. 20, the momentum gradient of the particle density equals

$$\begin{aligned}
\partial_p\langle f\,\rangle &= -f_0\bigg[\frac{\gamma}{p}\Big(1+\sum_A\xi^{(A)}\Big)-\sum_A\partial_p\xi^{(A)}\bigg],\\
\partial_\theta\langle f\,\rangle &= f_0\sum_A\partial_\theta\xi^{(A)},\\
\partial_\phi\langle f\,\rangle &= f_0\sum_A\partial_\phi\xi^{(A)}.
\end{aligned} \tag{21}$$

Observations indicate that the anisotropy, $\xi^{(A)}$, is independent of momentum, p. For example, Ahluwalia and Fikani (1996a,b) and el-Borie et al. (1996) show that there is no systematic rigidity dependence in the anisotropy over a wide range of neutron monitor cutoff rigidities, from 0 to 20 GV. Hence, $\sum\partial_p\xi^{(A)} = 0$ above.

3 Order of Magnitude Analysis

Further progress depends on an order of magnitude analysis of the integrand in Eq. 14. The order of magnitude of the term containing $\partial_p\langle f\rangle$ goes as

$$\delta B\,\beta_{SW}\gamma\frac{f_0(t')}{p(t')}\Big[1+\sum\xi^{(A)}(t')\Big], \tag{22}$$

where β_{SW} is the normalized solar wind. The order of magnitude of the term containing $\partial_\theta\langle f\rangle$ goes as

$$\delta B\,\Big[\beta_p+\beta_{SW}\Big]\frac{f_0(t')}{p(t')}\sum\xi^{(A)}(t'), \tag{23}$$

where β_p is the normalized particle speed. The order of magnitude of the term containing $\partial_\phi\langle f\rangle$ is similar to Eq. 23. Hence, the order of magnitude of the integrand in Eq. 14 goes as

$$\delta B\frac{f_0(t')}{p(t')}\Big[\gamma\beta_{SW}+(\gamma+2)\beta_{SW}\xi^{(A)}(t')+2\beta_p\xi^{(A)}(t')\Big]. \tag{24}$$

As previously stated, the coordinate dependence of $p(t')$, $f_0(t')$, and $\xi^{(A)}(t')$ is suppressed; it should be understood, for example, that $\xi^{(A)}(t') = \xi^{(A)}[\boldsymbol{x}(t')]$, where Eq. 10 contains expressions for $(\boldsymbol{x},p)$ in terms of t'. Keeping this in mind,

$$\xi^{(A)}(t') = \xi^{(A)}(x^\imath+\Delta^\imath) = \xi^{(A)}(x^\imath)\Big[1+\frac{\Delta^\jmath_x}{\ell^\jmath}\Big], \tag{25}$$

where $\Delta^\imath_x = x^\imath(t') - x^\imath(t)$ and $1/\ell^\jmath \equiv (\partial_{x^\jmath}\xi^{(A)})/\xi^{(A)}$ by definition. Observations (see, for example Zank et al., 1998), coupled with Fick's Law, indicate that $\ell\sim 10$ AU. Clearly, the Taylor series expansion of $\xi^{(A)}$ is only possible if $\Delta^\jmath_x/\ell^\jmath \ll 1$. This condition is fulfilled, because $\Delta^1_x\sim\Delta^2_x\sim R_L$, where R_L is the particle Larmor radius. Over the range of spacecraft and neutron monitor energies, $R_L < 0.1$ AU. In addition, during the time-frame of interest, $T = 0\to T_{CM}$, $\Delta^3_x\sim\lambda_C$, where $\lambda_C = 0.024$ AU is the magnetic correlation length of the solar wind.

In the same vein as Eq. 25,

$$\frac{f_0(t')}{p(t')} = \frac{f_0(t)}{p(t)}\bigg[1+\frac{\Delta^\imath_x}{L^\imath}\bigg]\bigg[1+\frac{\Delta_p}{p(t)}\bigg]^{-(\gamma+1)}, \tag{26}$$

where $\Delta_p = p(t') - p(t)$ and $1/L^\imath \equiv (\partial_{x^\imath}f_0)/f_0$ by definition. Webber and Lockwood (1999) report that $L > 20$ AU for >70 MeV cosmic rays observed by IMP, Voyager, and Pioneer spacecraft between 1978 and 1996. Chen and Bieber (1993) report that $L > 100$ AU based on neutron monitor data. Clearly, the Taylor series expansion of k is possible because $\Delta^\imath_x/L^\imath \ll 1$. Substituting Eq. 17 into Eq. 26 returns, to first-order in β,

$$\frac{f_0(t')}{p(t')} = \frac{f_0(t)}{p(t)}\bigg[1+\frac{\Delta^\imath_x}{L^\imath}\bigg]\bigg[1-\beta(\gamma+1)\Delta\mathcal{H}^{(1)}_p\bigg], \tag{27}$$

where $\Delta\mathcal{H}^{(1)}_p = \mathcal{H}^{(1)}_p(t') - \mathcal{H}^{(1)}_p(t)$. A cursory analysis of Eq. 17 indicates that the order of magnitude of $\Delta\mathcal{H}^{(1)}_p$ goes as $\sim 2/\beta_p$. Consequently, when $\gamma\sim 3$, the order of magnitude of Eq. 27 goes as

$$\frac{f_0(t')}{p(t')} \sim \frac{f_0}{p}\bigg(1+8\frac{\beta_{SW}}{\beta_p}+\frac{\Delta_x}{L}+8\frac{\Delta_x}{L}\frac{\beta_{SW}}{\beta_p}\bigg) \tag{28}$$

Now substitute Eq. 25 and Eq. 28 into Eq. 24; a careful order of magnitude analysis of all the terms involved, paying particular attention to the energy dependence of the terms, indicates that the order of magnitude of the integrand in Eq. 14 goes as

$$\delta B\frac{f_0(t)}{p(t)}\Big[\gamma\beta_{SW}+2\beta_p\xi^{(1)}(t)\Big], \tag{29}$$

subject to the constraint that the proton kinetic energy exceeds 10 MeV or, equivalently, that the proton rigidity exceeds 125 MV. The first term in Eq. 29 can be traced back to Eq. 22; that is, it is associated with the term containing $\partial_p\langle f\rangle$ and deals with convection. The second term in Eq. 29 can be traced back to Eq. 23; that is, it is associated with the term containing $\partial_\theta\langle f\rangle$ and deals with pitch angle scattering. Notice that the integrand in Eq. 14 is evaluated at time t', whereas the order of magnitude of Eq. 14 depends only on time t. Consequently, we can assume that f_0 and $\xi^{(1)}$ are constant over the time interval $T = 0 \rightarrow T_{CM}$; in particular, this indicates that f_0 and $\xi^{(1)}$ do not depend on position over the relevant time interval. Furthermore, notice that it is sufficient to use a first-order anisotropy approximation for the phase space density. Finally, notice that, subject to the above energy constraint, the unperturbed particle trajectory can be treated as if the electric field is not present.

4 Conclusion

In conclusion, an order of magnitude analysis indicates that

$$\Big\langle f(p,\theta,\phi)\Big\rangle = f_0(p)\Big[1 + \xi_1^{(1)}\sin\theta\cos\phi + \xi_2^{(1)}\sin\theta\sin\phi + \xi_3^{(1)}\cos\theta\Big], \qquad (30)$$

where $f_0(p) = k_0 p^{-\gamma}$ and $\xi^{(1)}$ are constants. Furthermore,

$$\begin{aligned} p(t) &= p_I, \\ \theta(t) &= \theta_I, \\ \phi(t) &= \Omega t - \phi_I. \end{aligned} \qquad (31)$$

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