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Anomalous transport of magnetic field lines in quasilinear regime: Analytical expressions

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Abstract. In weak magnetic turbulence, the diffusive prediction for the quasilinear spreading of magnetic field lines as a function of the distance Δz along the average field requires the existence of a sufficiently short correlation length L. Releasing the assumption concerning the existence of L, we present an analytical proof that, whenever the spectral index of the turbulence does not exactly vanish below the parallel wavenumber $10/\Delta z$, the transport of the field lines is anomalous (or non-diffusive) on the scale Δz. Simple expressions are derived for the transport exponent α and coefficient D_α (defined by a field line spreading equal to $D_\alpha \Delta z^\alpha$). This allows for a quantitative comparison with the prediction of the original quasilinear theory. Some consequences for the dispersion of solar particles in the interplanetary magnetic fields are also discussed.

1 Introduction

The transport of cosmic-ray particles across the regular component of the magnetic field, in both the interstellar and interplanetary media, is for a large part induced by the transport of the magnetic field lines themselves (Jokipii, 1966; Schlickeiser, 1994). At the shock fronts of supernovae like SN1987A, the observed acceleration time of GeV-electrons suggests a transport also dominated by the wandering of the magnetic field lines, as the inferred diffusion coefficient of the electrons by far exceeds the Bohm value of this coefficient (Ball and Kirk, 1992; Ragot, 2001a and b). Understanding the behavior of magnetic field lines in a turbulence composed of random fluctuations $\delta \boldsymbol{B}$ superimposed on a regular magnetic field $\boldsymbol{B}_0$ is thus of prime importance to model the propagation of charged particles in both astrophysical and space plasmas.

The case of small magnetic field perturbation is treated by the quasilinear theory (Jokipii and Parker, 1968) for weak magnetic turbulence. This theory, which neglects the perpendicular displacement of the field lines in the derivation of their spreading (first order derivation in $\delta b \equiv \delta B / B_0$), predicts a diffusion of the field lines beyond the parallel correlation length, $L_{c_\parallel}$, defined as the characteristic scale of the two-point correlation function. There is a strong belief among astrophysicists and physicists in general that, as long as the quasilinear approximation holds, *i.e.*, as long as the perpendicular displacement can be neglected, the quasilinear theory does predict a diffusion of the magnetic field lines or, more accurately, their linear spreading across the direction of $\boldsymbol{B}_0$ with the distance Δz along $\boldsymbol{B}_0$. However, this diffusive result is conditioned by the existence of a finite correlation length, $L_{c_\parallel}$, small enough to consider the transport of the field lines on much longer scales.

In the original papers by Jokipii and Parker (1968) and Jokipii and Coleman (1968), this correlation length was estimated as the inverse of the upper wavenumber in the low, flat part of the turbulence spectrum. A power spectrum flat below $k = L_c^{-1}$ produces indeed a correlation function of the magnetic field perturbation with an exponential cutoff of characteristic scale L_c. Yet a flattening of the spectrum at sufficiently high frequency is not guaranteed. For instance in the solar wind, the early observations apparently indicating a flattening at 10^{-5} Hz, which would have given a quasilinear correlation just short enough, have not been confirmed by more recent measurements which show power-law spectra down to lower frequencies (Goldstein et al., 1995). In general the presence of such extended, projected spectra, relatively smooth but not flat, is expected for an anisotropic turbulence (*e.g.*, Ragot, 1999a), and as the damping rates of many plasma waves depend on the propagation angle of the waves, anisotropic turbulence is likely to be a quite common feature of plasmas. Clearly, in those cases of extended projected spectra, a study of the field lines transport is still needed even in the quasilinear regime of magnetic field perturbation, as the spreading of the field lines on any relevant scale will be determined by a part of the spectrum that is not flat, hence neglected in the original quasilinear theory.

Here, we release the assumption concerning the existence of a short correlation length and express the spreading of the field lines along the axis x normal to the average magnetic field as a function of the projected power spectrum of turbulence. In the case when this projected spectrum can be

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described as a power law on an interval of wavenumbers around $1/\Delta z$, which is generally assumed in any study of turbulence, we then establish a new asymptotic expansion for the variance $\langle \Delta x^2 \rangle$. With this expansion, we analytically prove that whenever the spectral index of the turbulence does not exactly vanish on an interval of wavenumbers at least two or three decades broad around $1/\Delta z$, the transport of the field lines is non-diffusive, or anomalous: $\langle \Delta x^2 \rangle$ increases as $(\Delta z)^\alpha$ with α different from 1. This confirms the numerical result obtained by Ragot (1999) for similar power-law spectra. We also establish simple expressions for the transport exponent α, as well as the transport coefficient D_α, defined by $\langle \Delta x^2 \rangle = D_\alpha (\Delta z)^\alpha$. These expressions are particularly important for a quantitative comparison with the spreading predicted by the original quasilinear theory.

We consider here as in the paper by Ragot (1999a) a three-dimensional turbulence in quasilinear regime with a *continuous* spectrum; hence, unlike Pommois et al. (1999) we always keep the length scale Δz much shorter than the inverse of the minimum wavenumber, which is an absolute requisite to model a continuous spectrum. We start below by drawing the main lines of the classical quasilinear derivation.

2 Original quasilinear theory

In the quasilinear approximation, *i.e.*, if the perpendicular deviation is neglected, the displacement along the axis x of the field line that goes through the point $\boldsymbol{r}_0 = (x_0, y_0, z_0)$ can be written as

$$\Delta x = x(\boldsymbol{r}_0, z) - x_0 = \int_{z_0}^{z} b_x(x_0, y_0, z')dz' \ , \tag{1}$$

where $\boldsymbol{b}$ stands for $\delta\boldsymbol{B}/B_0$, and the variance $\langle \Delta x^2 \rangle$ can be expressed as:

$$\langle \Delta x^2 \rangle = \int_{z_0}^{z} dz' \int_{z_0}^{z} dz'' \, \langle b_x(x_0, y_0, z') b_x(x_0, y_0, z'') \rangle$$

$$= 2\Delta z \int_0^{\Delta z} ds \left(1 - \frac{s}{\Delta z}\right) R_{xx}(s) \ . \tag{2}$$

$\Delta z = z - z_0$. The brackets $\langle \ \rangle$ denote an average over a statistical ensemble of systems and $R_{xx}(s) = \langle b_x(x_0, y_0, z_0) \, b_x(x_0, y_0, z_0 + s) \rangle$ stands for the two-point correlation function of the magnetic field along x. In the usual quasilinear theory R_{xx} is assumed to cut off on the length scale $L_{c_\parallel}$, known as the parallel correlation length, and the limit $\Delta z \gg L_{c_\parallel}$ is taken so that

$$\frac{\langle \Delta x^2 \rangle}{2\Delta z} \approx \int_0^{+\infty} ds \, R_{xx}(s) \equiv D \ . \tag{3}$$

It shows that the magnetic field lines diffuse with the diffusion coefficient D on length scales much longer than $L_{c_\parallel}$. However, it does not prove that $L_{c_\parallel}$ exists and is very much smaller than the size of the system, which happens to be necessary to observe a diffusion *in* the system.

In the following, we release the assumption concerning the existence of a finite correlation length and derive a general expression for the spreading of magnetic field lines in the quasilinear regime of turbulence.

3 Quasilinear spreading of magnetic field lines

If k_m and k_M denote the lowest and highest wavenumbers in the spectrum, the spreading of the field lines:

$$\langle \Delta x^2 \rangle = 2k_m^3 \int_{z_0}^{z} dz' \int_{z_0}^{z} dz'' \int d\boldsymbol{k} \, b_x^2(\boldsymbol{k}) \cos[k_\parallel (z' - z'')] \tag{4}$$

can be deduced from

$$\frac{b_x(\boldsymbol{r})}{2} = \int d\boldsymbol{k}_\perp \int_0^{k_M} dk_\parallel \, \tilde{b}_x(\boldsymbol{k}) \cos\left(\boldsymbol{k}_\perp \cdot \boldsymbol{r} + k_\parallel z + \phi_{\boldsymbol{k}}\right) , \tag{5}$$

where $\tilde{b}_x(\boldsymbol{k}) e^{i\phi_{\boldsymbol{k}}}$ is the Fourier transform of $b_x(\boldsymbol{r})$ with $\tilde{b}_x(\boldsymbol{k}) > 0$. The derivation of Eq. (4) assumes, as in the quasilinear theory, that the phases $\phi_{\boldsymbol{k}}$ decorrelate on the scale k_m but this assumption of no spectral structuring could of course be released by introducing a different phase-correlation scale and substituting for the factor k_m^3. Integrating now over z' and z'', we obtain in the quasilinear regime of magnetic field perturbation:

$$\langle \Delta x^2 \rangle = 4k_m^3 \int_0^{k_M} dk_\parallel \, \left[1 - \cos\left(k_\parallel Z\right)\right] \frac{P_{x_\parallel}(k_\parallel)}{k_\parallel^2} \ , \tag{6}$$

where

$$P_{x_\parallel}(k_\parallel) = \int_0^{2\pi} d\phi \int_{k_{min}(k_\parallel)}^{k_{max}(k_\parallel)} dk_\perp \, k_\perp b_x^2(k_\parallel, k_\perp, \phi) \tag{7}$$

is the x-component of the power spectrum projected along $\boldsymbol{B}_0$. $k_{min}(k_\parallel) = [\max(0, k_m^2 - k_\parallel^2)]^{1/2}$ and $k_{max}(k_\parallel) = (k_M^2 - k_\parallel^2)^{1/2}$. A somewhat more detailed derivation of expression (6) can be found in Ragot (1999).

When the spectrum is smooth enough to be represented as a series of power laws, the right-hand side of Eq. (6) can be integrated over the parallel wavenumbers to obtain an explicit form of the field lines spreading.

For a power-law spectrum $P_{x_\parallel}(k_\parallel) = P_{x_\parallel}(k_1)(k_\parallel/k_1)^{-a}$ from k_1 to $+\infty$, we find in the quasilinear regime:

$$\langle \Delta x^2 \rangle = 4k_m^3 P_{x_\parallel}(k_1) k_1^{-1}$$

$$\times \left\{ \frac{1}{1+a} + |k_1 \Delta z|^{+1+a} \Gamma(-1-a) \sin\frac{a\pi}{2} - \right.$$

$$\left. \frac{1}{1+a} F_{P,Q} \left[\left\{ \frac{-1-a}{2} \right\}, \left\{ \frac{1}{2}, \frac{1-a}{2} \right\}; \frac{-(k_1 \Delta z)^2}{4} \right] \right\} \tag{8}$$

where $F_{P,Q}$ denotes the hypergeometric function and $a > -1$. When $a > -1$ and $k_1 \Delta z \ll 1$, an expansion of the hypergeometric function gives:

$$\langle \Delta x^2 \rangle = 4k_m^3 P_{x_\parallel}(k_1) k_1^{-1} \left\{ |k_1 \Delta z|^{1+a} \Gamma(-1-a) \sin\frac{a\pi}{2} \right.$$

$$-\frac{(k_1\Delta z)^2}{2(1-a)} + \bigcirc\left((k_1\Delta z)^4\right)\Bigg\} \ . \quad (9)$$

Eq. (9) shows that the spreading of the field lines is not linear unless $a = 0$. Moreover, since $\Gamma(-1-a)\sin(a\pi/2) \to \pi/2$ as $a \to 0$, the usual quasilinear diffusion coefficient $2\pi k_m^3 P_{x_\parallel}(k_1)$ is recovered in this limit of a flat spectrum.

For a finite, upper wavenumber k_2, one has to subtract $4k_m^3 P_{x_\parallel}(k_1)k_1^a \int_{k_2}^{+\infty} dk_\parallel \left[1-\cos\left(k_\parallel \Delta z\right)\right] k_\parallel^{-2-a}$ from the right-hand side of Eq. (8), which can be estimated in a similar way as the integral from k_1 to $+\infty$. However, the first term resulting from the integration of $k_\parallel^{-2-a}$ is small compared to the part in k_1 as soon as $(k_1/k_2)^{1+a} \ll 1$. As for the other term, it is negligible for $k_2\Delta z \gg 1$ and $a > -1$ because of the Riemann-Lebesgue lemna (Bender and Orszag, 1978), since $\int_{k_2}^{+\infty} dk_\parallel |k_\parallel|^{-2-a}$ exists. In consequence if $k_2^{-1} \ll \Delta z \ll k_1^{-1}$ and $a > -1$, the relations (8), (9) still apply for a power-law spectrum on a finite interval $[k_1, k_2]$.

When $a < -1$, or even approaches -1 from above, the upper boundary comes into play. The exponent α does not converge to a unique value but tends to 0 when averaged on a broad range of length scales Δz. As such spectral indexes are quite unrealistic for a turbulence spectrum, we do not discuss this point in more details.

4 Transport exponent and coefficient

Our new formulation for the variance $\langle \Delta x^2 \rangle$ (Eq. [9]) presents a real advantage over the one of Ragot (1999). The transport exponent α and transport coefficient D_{m_α}, defined by

$$\langle \Delta x^2 \rangle \approx D_{m_\alpha} (\Delta z)^\alpha \ , \quad (10)$$

can now be expressed in analytical form. From $\alpha = d(\log \langle \Delta x^2 \rangle)/d(\log \Delta z)$, we obtain for spectral indexes $a > -1$ or -0.5 (depending on how small $k_1\Delta z$ is),

$$\alpha = 1 + \frac{aA_1(k_1\Delta z)^{1+a} + A_2(k_1\Delta z)^2}{A_1(k_1\Delta z)^{1+a} + A_2(k_1\Delta z)^2} \quad (11)$$

with $A_1 = \Gamma(-1-a)\sin(a\pi/2)$ and $A_2 = -1/[2(1-a)]$. In the limit of small $|a|$, namely, $|a| < 0.5$ for $k_1\Delta z \sim 10^{-1}$ and $|a| < 1$ for $k_1\Delta z \to 0$, we also have

$$D_{m_\alpha} = 4k_m^3 P_{x_\parallel}(k_1)k_1^a A_1 \ , \quad (12)$$

whereas in the limit of larger $|a|$ ($|a| > 2$ for $k_1\Delta z \sim 10^{-1}$ and $|a| > 1$ for $k_1\Delta z \to 0$),

$$D_{m_\alpha} = 4k_m^3 P_{x_\parallel}(k_1)k_1^a A_2 \ . \quad (13)$$

Figure 1 shows the transport exponent α as a function of the spectral index a. The transport of the magnetic field lines is supradiffusive ($\alpha > 1$) for any positive spectral index of turbulence and subdiffusive ($\alpha < 1$) for any inverted spectrum, which confirms the results of Ragot (1999). Moreover, $|\alpha - 1|$ already exceeds 0.5 for a spectral index of 0.6, in absolute value. These values of the transport exponent can also easily be guessed from the expansion (9).

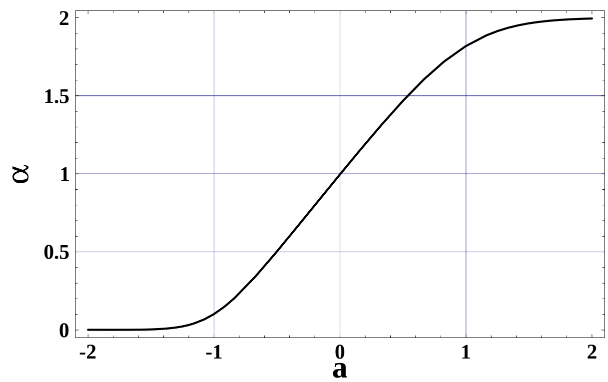

Fig. 1. Transport exponent α as a function of the spectral index a for $k_1\Delta z = 10^{-2}$. The spreading of magnetic field lines in the quasilinear regime of turbulence is only linear for a flat spectrum. For all decreasing power laws ($a > 0$), the field lines supradiffuse ($\alpha > 1$), whereas for inverted power laws, the field lines subdiffuse ($\alpha < 1$), as long as $a > -2$. For $a < -0.5$, α is averaged on a broad range of Δz.

As the spectral index a approaches 1 from below, the term in $-(k_1\Delta z)^2/[2(1-a)]$ has a growing weight in Eq. (9), due to the factor $1/(1-a)$. It is dominant for $a > 1$ so that its sum with $(k_1\Delta z)^{1+a}\Gamma(-1-a)\sin(a\pi/2)$, which is then negative, remains always positive. In the limit of very small $k_1\Delta z$, the other term becomes completely negligible and α converges to 2 as soon as $a > 1$. For $k_1\Delta z = 10^{-2}$ as in Figure 1, the first term still has a significant weight up to $a = 1.5 - 2$, but for all spectral indexes steeper than 2, the transport exponent α is practically equal to 2.

For $|a| < 0.5 - 0.8$, $\langle \Delta x^2 \rangle$ is accurately determined by the first term. In this range of spectral indexes, the transport exponent simply reduces to $\alpha = 1 + a$. This case is of particular interest since it corresponds to a spectrum that would tend to flatten at low frequency, but not perfectly, as is observed for instance in the solar wind (Goldstein et al., 1995) (see Fig. 2, Ragot, 1999b).

Note that $P_{x_\parallel}(k_1)k_1^a = P_{x_\parallel}(k_\parallel)k_\parallel^a$ for any $k_\parallel$ in the interval $[k_1, k_2]$ so that the value of D_{m_α} does not depend on the lower limit k_1 of the interval on which $P_{x_\parallel}$ is in $k_\parallel^{-a}$, but solely on the level of turbulence in this interval of parallel wavenumbers. The range of validity in a for $\alpha = 1 + a$, however, does depend on the value of $k_1\Delta z$. If $k_1\Delta z \to 0$, it extends from -1 to 1.

The condition established by Ragot (1999a) to observe, in the quasilinear regime of turbulence, a diffusive spreading of the field lines on at least one decade is confirmed; namely, the spectrum should be flat on at least three decades around $1/\Delta z$ (2 decades for $k_2^{-1} \ll \Delta z \ll k_1^{-1}$ plus 1 for the variation of Δz). This means that even a flattening at 10^{-5} Hz in the solar wind would not have guaranteed a diffusive spreading of the field lines on a length scale shorter than the typical distance between strong inhomogeneities, since the sun rotates at a frequency of $3.2 - 4.6 \times 10^{-7}$ Hz less than 100 times smaller. This conclusion of no quasilinear diffusion

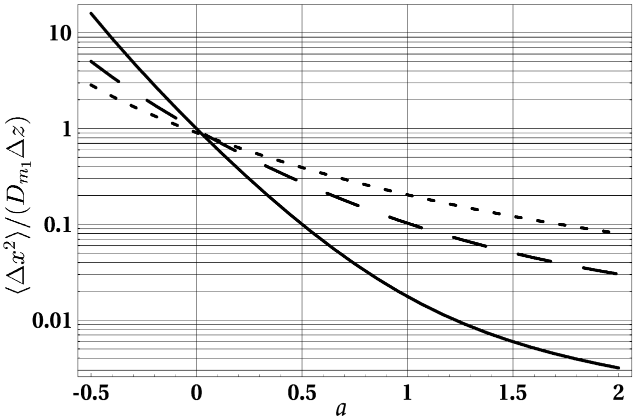

Fig. 2. Ratio of the quasilinear field line spreading over the original, diffusive quasilinear prediction for a given value of $P_{x_\parallel}(k_1)$. Continuous line: $k_1\Delta z = 0.01$. Long-dashed line: $k_1\Delta z = 0.1$. Short-dashed line: $k_1\Delta z = 0.3$.

of magnetic field lines in the inner heliosphere is not contradicted by the observation of fluctuating field line directions and could account for the lack of mixing of charged particles propagating through the turbulent solar wind (Zurbuchen et al., 2000) recently observed with the SWICS instrument on ACE. A supradiffusion is indeed characterized by a lower dispersion and ordered fields on many scales (see below).

5 Comparison with the original quasilinear prediction

A quantitative comparison of our prediction for the magnetic field line spreading with the prediction of the original quasilinear theory is relatively straightforward by now. For a same level of turbulence at the lower wavenumber k_1, the ratio of the two predicted variances can be written as

$$\frac{\langle\Delta x^2\rangle}{D_{m_1}\Delta z} = \frac{2}{\pi}A_1(k_1\Delta z)^a\left[1+\frac{A_1}{A_2}(k_1\Delta z)^{1-a}\right] + \bigcirc\left((k_1\Delta z)^3\right) \quad . \tag{14}$$

We plotted this ratio in Figure 2 for various values of $k_1\Delta z$. What strikes at once is that the supradiffusion ($a > 0$) seems to give a much slower spreading of the field lines than would be expected for the diffusion of the original quasilinear theory, whereas the subdiffusion apparently gives a much faster transport. While this might not be entirely accurate (one chooses a lower turbulence amplitude for larger a by taking the same value of $P_{x_\parallel}[k_1]$), it serves our purpose pretty well here. We want indeed to emphasize the following. A supradiffusion does not necessarily mean that the transport is faster, nor does a subdiffusion imply a slower transport. This very much depends on the value of the transport coefficient.

Supradiffusion is characterized by a lesser dispersion of the field lines which tend to behave in a more orderly manner. While the small-scale irregularities still exist and might give the impression that the field lines are "diffusing" in an erratic and uncorrelated way, the large-scale transport is significantly influenced by the lower part of the spectrum and ordered behavior occurs on all scales, even the largest ones. The propagator derived by Ragot and Kirk (1997) illustrates this property with a peaked shape shifted away from zero. For comparison, the propagator of diffusion is the well-known Gaussian centered around the origin. In the subdiffusive case ($a < 0$), the propagator is more widespread and peaks at the origin. The transport is dominated by the small scales and long ordered "flights" are extremely rare. A greater dispersion might still result, though, from the fact that some field lines can wander relatively quickly in some part of the space while others (the majority) are trapped in smaller-scale domains on longer length scales.

6 Conclusion

To summarize, we have shown analytically that in the quasilinear regime of turbulence, the transport of magnetic field lines is anomalous on the length scale Δz whenever the projected spectrum of turbulence is not perfectly flat below the parallel wavenumber $10/\Delta z$. The field line spreading $\langle\Delta x^2\rangle$ varies as $(\Delta z)^\alpha$ with $\alpha \neq 1$, $0 \leq \alpha \leq 2$. A decreasing spectrum results in a supradiffusion of the field lines ($\alpha > 1$), whereas an inverted spectrum implies a subdiffusion ($\alpha < 1$). For a spectrum that takes the form of a power-law on an interval of parallel wavenumbers around $(\Delta z)^{-1}$, we established new, simple expressions for the transport exponent and coefficient (Eq. [11], [13]). These expressions generalize the quasilinear prediction for the spreading of magnetic field lines.

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