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Anomalous diffusion of the cosmic rays: steady state solution

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Abstract.
We consider the propagation of galactic cosmic rays in the fractal interstellar medium. Steady state solution of the fractional diffusion equation, describing cosmic ray propagation, is found. We show that the exponent of the steady state spectrum turns out to be equal to the exponent of the cosmic ray spectrum above the "knee", that is $\eta \approx 3.1$.

1 Introduction

The steady state diffusion equation is frequently used for interpretation of the cosmic ray phenomena (see, for example,Ginzburg and Syrovatskii (1964); Berezinsky et al. (1990); Blumen et al. (1991); Zirakashvili et al. (1991); Ptuskin et al. (1993); Kalmykov et al (1999)). Without energy losses and nuclear interactions this equation for concentration of the cosmic rays with energy E generated by sources distribution with density function $S(\boldsymbol{r}, E)$ has the form:

$$D(E)\triangle N(\boldsymbol{r}, E) + S(\boldsymbol{r}, E) = 0, \qquad (1)$$

where $D(E)$ is a diffusivity. The equation (1) describes the cosmic ray propagation under the assumption that nonhomogeneities of medium have small-scale character. If the medium has a fractal structure, the normal diffusion equation (1) is not a proper one. A generalization of this equation leads to an anomalous diffusion (see the reviews by Bouchaud and Georges (1990); Isichenko (1992); West end Deering (1994); Uchaikin and Zolotarev (1999) and paper Lagutin and Uchaikin (2001)). In recent papers (Lagutin et al., 2000, 2001; Lagutin and Uchaikin, 2001) the anomalous diffusion equations under the different approximations were formulated and the solutions for point instantaneous and impulse sources were found.

The main goal of this paper is to find the steady state solution of the fractional diffusion equation and to evaluate the exponent of steady state energy spectrum.

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2 Fractional diffusion equation

The fractional diffusion equation formulated in (Lagutin and Uchaikin, 2001) has the form :

$$\frac{\partial N(\boldsymbol{r}, E, t)}{\partial t} = -D(E, \alpha, \beta) D_{0+}^{1-\beta}(-\triangle)^{\alpha/2} N(\boldsymbol{r}, E, t) + S(\boldsymbol{r}, E, t), \qquad (2)$$

here $D(E, \alpha, \beta) = D_0(\alpha, \beta)E^{\delta}$ is the anomalous diffusivity, α, β are determined by the fractal structure of the medium and by trapping mechanism, correspondingly. D_{0+}^{μ} means fractional derivative of Rieman-Liouville by time (Samko et al., 1987):

$$D_{0+}^{\mu} f \equiv \frac{1}{\Gamma(1-\mu)} \frac{d}{dt} \int_0^t (t-\tau)^{-\mu} f(\tau) d\tau, \quad \mu < 1,$$

$(-\triangle)^{\alpha/2}$ is fractional Laplacian (called "Riss' operator") (Samko et al., 1987):

$$(-\triangle)^{\alpha/2} f(x) = \frac{1}{d_{m,l}(\nu)} \int_{R^m} \frac{\triangle_y^l f(x)}{|y|^{m+\nu}} dy,$$

where $l > \alpha$, $x \in \mathrm{R}^m$, $y \in \mathrm{R}^m$,

$$\Delta_y^l f(x) = \sum_{k=0}^{l} (-1)^k \binom{l}{k} f(x - ky)$$

$$d_{m,l}(\nu) = \int_{\mathrm{R}^m} (1 - e^{iy})^l |y|^{-m-\nu} dy,$$

If $\alpha = 2$ and $\beta = 1$, the equation (2) is the normal diffusion equation. If $\alpha < 2$, $\beta = 1$, we have from (2) the superdiffusion equation:

$$\frac{\partial N(\boldsymbol{r},E,t)}{\partial t} = -D(E,\alpha)(-\triangle)^{\alpha/2}N(\boldsymbol{r},E,t) + S(\boldsymbol{r},E,t), \quad (3)$$

discussed in Lagutin et al. (2001).

3 Steady state solution of superdiffusion equation

In steady state case the equation (3) takes the form:

$$D(-\triangle)^{\alpha/2}N(\boldsymbol{r},E) = S(\boldsymbol{r},E). \quad (4)$$

The Green's function $G(\boldsymbol{r},E;E_0)$ satisfies the equation:

$$D(-\triangle)^{\alpha/2}G(\boldsymbol{r},E;E_0) = \delta(E-E_0)\delta(\boldsymbol{r}). \quad (5)$$

The solution of the equation (5) can be found by means of Fourier transformation:

$$\widetilde{f}(\boldsymbol{k}) = \widehat{F}f(\boldsymbol{r}) = \int\limits_{R^3} e^{i\boldsymbol{kr}}f(\boldsymbol{r})d\boldsymbol{r}.$$

Taking into account that the Fourier transform of fractional Laplacian is (Samko et al., 1987)

$$\widehat{F}(-\triangle)^{\alpha/2}G(\boldsymbol{r},E;E_0) = |\boldsymbol{k}|^\alpha \widetilde{G}(\boldsymbol{k},E;E_0),$$

it's easy to find:

$$\widetilde{G}(\boldsymbol{k},E;E_0) = \frac{\delta(E-E_0)}{D|\boldsymbol{k}|^\alpha} = \delta(E-E_0)\int\limits_0^\infty dy e^{-D|\boldsymbol{k}|^\alpha y}.$$

Applying the inverse Fourier transformation, we obtain:

$$G(\boldsymbol{r},E;E_0) = \frac{\delta(E-E_0)}{(2\pi)^3}\int\limits_0^\infty dy \int\limits_{R^3} d\boldsymbol{k} e^{-i\boldsymbol{kr}-D|\boldsymbol{k}|^\alpha y}.$$

Since

$$g_3^{(\alpha)}(r) = \frac{1}{(2\pi)^3}\int\limits_{R^3} d\boldsymbol{k} e^{-i\boldsymbol{kr}-|\boldsymbol{k}|^\alpha}$$

is density of three-dimensional spherically-symmetrical stable law, for Green's function we have:

$$G(\boldsymbol{r},E;E_0) = \frac{\delta(E-E_0)}{(D)^{3/\alpha}}\int\limits_0^\infty dy y^{-3/\alpha} g_3^{(\alpha)}\left(\frac{|\boldsymbol{r}|}{(Dy)^{1/\alpha}}\right). \quad (6)$$

Using Green's function (6) we can find the solution of equation (4) for a interesting for astrophysics source. Thus, for point source with inverse power spectrum relating to supernova burst $S(\boldsymbol{r},E) = S_0E^{-p}\delta(\boldsymbol{r})$, the solution of the equation has the form:

$$N(\boldsymbol{r},E) = \frac{S_0E^{-p}}{(D)^{3/\alpha}}\int\limits_0^\infty dy y^{-3/\alpha} g_3^{(\alpha)}\left(\frac{|\boldsymbol{r}|}{(Dy)^{1/\alpha}}\right) \quad (7)$$

Taking into account Mellin transform of spherically-symmetrical three-dimensional stable laws (Uchaikin and Zolotarev, 1999):

$$g_m^{(\alpha)}(s) = \int\limits_0^\infty g_m^{(\alpha)}(r)r^{s-1}dr = \frac{2^s\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{m-s}{\alpha}\right)}{\alpha(4\pi)^{m/2}\Gamma\left(\frac{m-s}{2}\right)}, \quad (8)$$

we have

$$N(\boldsymbol{r},E) = \frac{2^{-\alpha}S_0}{\pi^{3/2}D_0r^{3-\alpha}}\frac{\Gamma\left(\frac{3-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)}E^{-p-\delta}. \quad (9)$$

To evaluate numerically the value of spectral exponent $\eta = p+\delta$, let's consider in details a passage from non-stationary solution to steady state one in a special case $\alpha = 1$. This choice is due to the fact that $g_3^{(1)}$ has the analytical representation. It's three-dimensional Cauchy's density:

$$g_3^{(1)} = \frac{1}{\pi^2(1+r^2)^2}.$$

Based on the results obtained in Lagutin et al. (2001) for points impulse source

$$S(\boldsymbol{r},t,E) = S_0E^{-p}\delta(\boldsymbol{r})\Theta(T-t)\Theta(t),$$

in our case we have

$$N(\boldsymbol{r},T,E) = \frac{S_0E^{-p-\delta}}{2\pi D_0r^2T}\left[1-\frac{1}{\frac{D_0^2T^2}{r^2}E^{2\delta}+1}\right]. \quad (10)$$

Using the representation $N(\boldsymbol{r},E) = N_0E^{-\eta}$ from (10), one can easily find the spectral exponent:

$$\eta(T) = p+\delta-\frac{2\delta}{\left(\frac{TD_0E^\delta}{r}+1\right)} \quad (11)$$

It should be noted that the solution (10) has the "knee" at $E_0(T) = \left(\frac{r}{TD_0}\right)^{1/\delta}$. At the "knee" energy E_0 the spectral exponent for observed particles η is equal to the spectral exponent for particles generated by the sources:

$$\eta = p.$$

One can also see from (11) that at $E \ll E_0$ and $E \gg E_0$, we have correspondingly:

$$\eta_{E\ll E_0} \approx p-\delta,$$

$$\eta_{E\gg E_0} \approx p+\delta, \quad (12)$$

that is the spectrum steepening on 2δ. The steady state solution is connected with $N(\boldsymbol{r},T,E)$ by means of passage to the limit:

$$N(\boldsymbol{r},E) = \lim_{T\to\infty} N(\boldsymbol{r},T,E). \quad (13)$$

So, we have

$$\eta = \lim_{T\to\infty}\eta(T) = p+\delta. \quad (14)$$

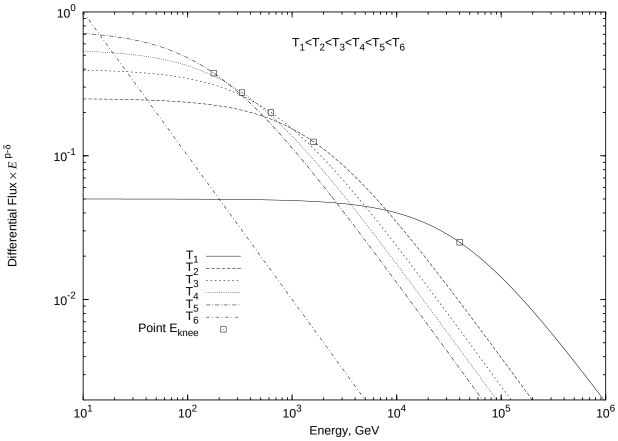

Fig. 1. Modification of the energy spectrum versus T ($T_6 \to \infty$). Square-position of the "knee"

$$E_0 = \lim_{T\to\infty} E_0(T) \to 0. \qquad (15)$$

Thus, the analysis presented above shows that the exponent of the steady state spectrum turns out to be equal to the spectral exponent above the "knee", that is $p + \delta \approx 3.1$. This conclusion is illustrated by (fig.1).

4 Steady state solution of fractional diffusion equation($\alpha < 2, \beta < 1$)

It has been shown in (Lagutin and Uchaikin, 2001) that the solution of equation (2) for the point impulse source has the form:

$$N(\boldsymbol{r}, t, E) = \frac{S_0 E^{-p}}{D(E,\alpha,\beta)^{3/\alpha}} \int\limits_{max[0,t-T]}^{t} \tau^{-3\beta/\alpha} \times \Psi_3^{(\alpha,\beta)}\left(|\boldsymbol{r}|(D(E,\alpha,\beta)\tau^{\beta})^{-1/\alpha}\right), \qquad (16)$$

where the scaling function $\Psi_3^{(\alpha,\beta)}(r)$,

$$\Psi_3^{(\alpha,\beta)}(r) = \int\limits_0^\infty q_3^{(\alpha)}(r\tau^\beta) q_1^{(1,\beta)}(\tau)\tau^{3\beta/\alpha} d\tau, \qquad (17)$$

is determined by three-dimensional spherically-symmetrical stable distribution $q_3^{(\alpha)}(r)$ and one-sided stable distribution $q_1^{(1,\beta)}(t)$ with characteristic exponent $\beta < 1$ (Uchaikin and Zolotarev, 1999):

$$q_1^{(\beta,1)}(t) = (2\pi i)^{-1} \int\limits_{\gamma-i\infty}^{\gamma+i\infty} e^{(\lambda t - \lambda^\beta)} d\lambda. \qquad (18)$$

As the integral diverges if $T \to \infty$, the spectral exponent has been evaluated for $T \sim 10^{10}y$. We found $\eta \approx p + \frac{\delta}{\beta}$.

5 Conclusion

We have considered the propagation of galactic cosmic rays in the fractal interstellar medium. Steady state solution of the fractional diffusion equations describing cosmic ray propagation have been found. We have shown that the exponent of the steady state spectrum turns out to be equal to the exponent of cosmic ray spectrum above the "knee", that is $\eta \approx 3.1$.

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