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The BGK Boltzmann equation and anisotropic diffusion

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Abstract. In this paper, we study a model of cosmic ray diffusion based on a gyro-phase, and pitch-angle dependent BGK Boltzmann model, involving two collision time scales $\tau_\perp$ and $\tau_\|$ associated with scattering perpendicular and parallel to the background magnetic field $\mathbf{B}_0$. The time scale $\tau_\perp$ describes the ironing out of gyro-phase anisotropies, and the relaxation of the full gyro-phase distribution f to the gyro-averaged distribution f_0. The time scale $\tau_\perp$ determines the diffusion coefficient $\kappa_\perp$, perpendicular to the mean magnetic field, and the corresponding anti-symmetric diffusion coefficient κ_A associated with particle drifts. The time scale $\tau_\|$ describes the relaxation of the pitch angle distribution f_0 to the isotropic distribution F_0, and determines the parallel diffusion coefficient $\kappa_\|$. The Green function solution of the model equation is obtained, for the case of delta function initial data in position, pitch angle and gyro-phase, in terms of Fourier-Laplace transforms. The solutions are used to discuss non-diffusive and diffusive particle transport. The gyro-phase dependent solutions exhibit cyclotron resonant behaviour, modified by resonance broadening due to $\tau_\perp$.

1 Introduction

Early work by Parker (1965) and Axford (1965) derived the form of the diffusion tensor for cosmic rays in a random magnetic field, for the case of isotropic scattering. Forman et al. (1974) used quasi-linear theory in slab turbulence to determine the diffusion coefficients parallel ($\kappa_\|$) and perpendicular ($\kappa_\perp$) to the mean magnetic field $\mathbf{B}_0$, as well as the anti-symmetric component of the diffusion tensor, κ_A, associated with particle drifts, for the case where the distribution function could be expanded in spherical harmonics. Jokipii (1971) and Hasselmann and Wibberenz (1970), pointed out that the detailed dependence of the pitch angle diffusion coefficient $D_{\mu\mu}$ on μ is important in determining $\kappa_\|$.

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2 The Model

The BGK Boltzmann equation for the momentum space distribution function $f(\mathbf{r},\mathbf{p},t)$, for particles with momentum $\mathbf{p}$, (velocity $\mathbf{v}$), at position $\mathbf{r}$ at time t, in a uniform background magnetic field $\mathbf{B}_0 = (0,0,B_0)^T$ along the z-axis, may be written in the form:

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla f - \Omega\frac{\partial f}{\partial \phi} = -\left(\frac{f-f_0}{\tau_\perp} + \frac{f_0-F_0}{\tau_\|}\right), \qquad (1)$$

where

$$f_0 = \frac{1}{2\pi}\int_0^{2\pi} f\,d\phi, \quad F_0 = \frac{1}{2}\int_{-1}^{1} f_0\,d\mu, \qquad (2)$$

denote the gyro-phase averaged distribution function (f_0), and the isotropic component of the distribution function (F_0) in momentum space, and $\mu=\cos\theta$ is the pitch angle cosine. The gyro-phase derivative term $-\Omega f_\phi$, on the left-hand side of (1), is the Lorentz force term, where $\Omega = qB_0/(mc)$ is the particle gyro-frequency, and m is the relativistic particle mass. Note that (v,θ,ϕ) are spherical polar coodinates for the velocity, where the polar axis is along $\mathbf{B}_0$. Kota (1993) used a model similar to (1), except that he used a pitch angle and gyro-phase diffusion term for the collision term.

3 The Diffusion Approximation

Following the approach of Kota (1993), we expand the distribution function in the series:

$$f = \sum_{n=-\infty}^{\infty} f_n \exp(in\phi), \qquad (3)$$

where$f_{-n} = f_n^*$. Multiplying the Boltzmann equation by $\exp(-im\phi)$, and integrating over the gyro-phase ϕ from $\phi = 0$ to $\phi = 2\pi$, yields the moment equations:

$$\frac{\partial f_m}{\partial t} + \frac{v\sin\theta}{2}\left(\frac{\partial f_{m-1}}{\partial x} - i\frac{\partial f_{m-1}}{\partial y}\right)$$

$$+\frac{v\sin\theta}{2}\left(\frac{\partial f_{m+1}}{\partial x}+i\frac{\partial f_{m+1}}{\partial y}\right)$$
$$+v\cos\theta\frac{\partial f_m}{\partial z}-im\Omega f_m$$
$$=-[(f_m-f_0\delta_0^m)/\tau_\perp+(f_0-F_0)\delta_0^m/\tau_\parallel], \quad (4)$$

where $m=0,\pm1,\pm2,\ldots$, and δ_j^i is the Kronecker delta symbol. In particular, for $m=0$, (4) multiplied by $2\pi p^2\sin\theta$, and integrated over θ from $\theta=0$ to $\theta=\pi$, yields the number density conservation equation:

$$N_t+\nabla\cdot\mathbf{S}=0, \quad (5)$$

where $N=p^2\int f d\Omega$ and $\mathbf{S}=p^2\int \mathbf{v} f d\Omega$ are the particle number density and current, and the integrations over $d\Omega$ are over solid angle in momentum space. In the diffusion approximation, one uses the approximate moment balance equations for $m=0$ and $m=1$:

$$2\pi p^2 v\int_0^\pi d\theta\sin\theta\cos\theta\left(v\cos\theta\frac{\partial f_0}{\partial z}+\frac{f_0-F_0}{\tau_\parallel}\right)\approx 0, \quad (6)$$

$$2\pi p^2 v\int_0^\pi d\theta\sin^2\theta\left(\frac{v\sin\theta}{2}\left(\frac{\partial f_0}{\partial x}-i\frac{\partial f_0}{\partial y}\right)\right.$$
$$\left.-i\Omega f_1+\frac{f_1}{\tau_\perp}\right)\approx 0, \quad (7)$$

to determine the diffusive current $\mathbf{S}$. The diffusion approximation, assumes that the scattering is strong enough to drive the distribution function to a near isotropic state, and that the effective scattering time is much shorter than the time scale for the evolution of F_0. Using (6) and (7), it follows that the diffusive current has the form:

$$\mathbf{S}=-\kappa_\parallel N_z\mathbf{e}_B-\kappa_\perp\nabla_\perp N-\kappa_A\nabla N\times\mathbf{e}_B, \quad (8)$$

where $\mathbf{e}_B\equiv\mathbf{e}_z$ is the unit vector along $\mathbf{B}_0$, and

$$\kappa_\parallel=\frac{v^2\tau_\parallel}{3},\quad \kappa_\perp=\frac{v^2\tau_\perp}{3(1+\Omega^2\tau_\perp^2)},\quad \kappa_A=\Omega\tau_\perp\kappa_\perp. \quad (9)$$

The expressons (9) for $\kappa_\parallel$, $\kappa_\perp$ and κ_A, have the same form as Forman et al. (1974).

4 The Green Function

Introducing the Laplace-Fourier transform:

$$\tilde f(\mathbf{k},\mathbf{p},s) = \int_0^\infty dt\int_{-\infty}^\infty\frac{d^3r}{(2\pi)^3}\exp(-st-i\mathbf{k}\cdot\mathbf{r})$$
$$f(\mathbf{r},\mathbf{p},t), \quad (10)$$

the BGK Boltzmann equation (1) reduces to the ordinary differential equation:

$$\Omega\tilde f_\phi-(s+i\mathbf{k}\cdot\mathbf{v}+\nu_\perp)\tilde f=$$
$$[-\hat f(\mathbf{k},\mathbf{p},0)+(\nu_\parallel-\nu_\perp)\tilde f_0-\nu_\parallel\tilde F_0], \quad (11)$$

where $\nu_\parallel=1/\tau_\parallel$, $\nu_\perp=1/\tau_\perp$ and $\hat f(\mathbf{k},\mathbf{p},0)$ is the Fourier transform of the initial data $f(\mathbf{r},\mathbf{p},0)$. For Dirac-delta initial data, with $f(\mathbf{r},\mathbf{p},0)=A\delta(\mathbf{r}-\mathbf{r}_0)\delta(\mu-\mu_0)\delta(\phi-\phi_0)$, we obtain

$$\hat f(\mathbf{k},\mathbf{p},0) = [A/(2\pi)^3]\exp(-i\mathbf{k}\cdot\mathbf{r}_0)$$
$$\delta(\mu-\mu_0)\delta(\phi-\phi_0). \quad (12)$$

Using (12) as the source term in (11), and integrating (11) yields the solution:

$$\tilde f = [\Omega I(\phi,\theta)]^{-1}\{(\nu_\parallel\tilde F_0-(\nu_\parallel-\nu_\perp)\tilde f_0)\times$$
$$\times[\mathcal{I}(2\pi,\theta)/\zeta-\mathcal{I}(\phi,\theta)]\}+Q, \quad (13)$$

where

$$Q = \frac{AI(\phi_0,\theta_0)\exp(-i\mathbf{k}\cdot\mathbf{r}_0)\delta(\mu-\mu_0)}{(2\pi)^3\Omega I(\phi,\theta)}\times$$
$$\times\left(\frac{1}{\zeta}-H(\phi-\phi_0)\right) \quad (14)$$

is the source term associated with the initial data (12). In deriving (13) and (14), the angles ϕ and ϕ_0 are restricted to the range $[0,2\pi]$, and the condition $\tilde f(\phi=0)=\tilde f(\phi=2\pi)$ is used to determine the integration constant. In (13) and (14)

$$I(\phi,\theta) = \exp\Big(-(\bar s+\bar\nu_\perp+ik_\parallel r_g\cos\theta)(\phi-\Phi)$$
$$-ik_\perp r_g\sin\theta\sin(\phi-\Phi)\Big), \quad (15)$$

is the integrating factor for (11), where we use the notation $\bar s=s/\Omega$, $\bar\nu_\perp=\nu_\perp/\Omega$, $\bar\nu_\parallel=\nu_\parallel/\Omega$ and

$$\mathcal{I}(\theta,\phi)=\int_0^\phi I(\phi',\theta)d\phi',$$
$$\zeta=1-\exp[-2\pi(\bar\nu_\perp+\bar s+ik_\parallel r_g\cos\theta)]. \quad (16)$$

In (14)-(16), (k,Θ,Φ) are spherical polar coordinates for $\mathbf{k}$, with polar axis along $\mathbf{B}_0$; $k_\parallel=k\cos\Theta$ and $k_\perp=k\sin\Theta$ and $H(x)$ is the Heaviside step function. Equation (13) can be regarded as an integral equation for $\tilde f$, and is a central result in the analysis.

By using the standard generating function identity for Bessel functions (e.g. Abramowitz and Stegun, 1965, p. 361, formula 9.1.41) we obtain

$$\mathcal{I}(\phi,\theta)=\sum_{n=-\infty}^{\infty}\frac{J_n(k_\perp r_g\sin\theta)}{\bar s+\bar\nu_\perp+i(k_\parallel r_g\cos\theta+n)}\times$$
$$\times\exp[(\bar s+\bar\nu_\perp+ik_\parallel r_g\cos\theta+in)\Phi]\times$$
$$\times\{1-\exp(-[\bar s+\bar\nu_\perp+ik_\parallel r_g\cos\theta+in]\phi)\}, \quad (17)$$

for $\mathcal{I}(\phi,\theta)$ where $r_g=pc/(qB_0)$ is the particle gyro-radius and $J_n(x)$ is a Bessel function of the first kind of order n and argument x. By noting that $\bar s=s/\Omega$, and setting $s=-i\omega$, one finds that the denominator of the n^{th} term in (17), $\bar s+\bar\nu_\perp+i(k_\parallel r_g\cos\theta+n)=0$ when

$$\omega-k_\parallel v\mu=n\Omega-i\nu_\perp,\quad \text{(n integer)}, \quad (18)$$

where $\mu=\cos\theta$. Thus the pole for the term indexed by n in the series (17) corresponds to the cyclotron resonance condition $\omega - k_{\|} v\mu = n\Omega$ broadened by scattering due to $\nu_{\perp}$.

Averaging (13) over gyro-phase ϕ yields the integral equation

$$\tilde{f}_0 = (a\tilde{F}_0 + \bar{Q})/[1 + (1 - \tau_{\|}/\tau_{\perp})a], \tag{19}$$

relating $\tilde{f}_0$ and $\tilde{F}_0$ where $\bar{Q} = \int_0^{2\pi} Q d\phi/(2\pi)$. The function a in (19) can be expressed in the form:

$$a = \bar{\nu}_{\|} \sum_{n=-\infty}^{\infty} \frac{J_n^2(k_{\perp} r_g \sin\theta)}{\bar{s} + \bar{\nu}_{\perp} + i(k_{\|} r_g \cos\theta + n)}. \tag{20}$$

The source term $\bar{Q}$ in (19) can be expressed in the form:

$$\bar{Q} = -\frac{AI(\phi_0, \theta_0)\exp(-i\mathbf{k}\cdot\mathbf{r}_0)}{(2\pi)^4\Omega}\hat{Q}, \tag{21}$$

where

$$\begin{aligned}\hat{Q} &= \exp[-(\bar{s} + \bar{\nu}_{\perp} + ik_{\|} r_g \cos\theta)] \times \\ &\times \sum_{n=-\infty}^{\infty} \frac{J_n(k_{\perp} r_g \sin\theta)\exp(-in\Phi)}{\bar{s} + \bar{\nu}_{\perp} + i(k_{\|} r_g \cos\theta + n)} \times \\ &\times \{1 + \exp[2\pi(\bar{s} + \bar{\nu}_{\perp} + ik_{\|} r_g \cos\theta)] \\ &\quad - \exp[\phi_0(\bar{s} + \bar{\nu}_{\perp} + ik_{\|} r_g \cos\theta)]\}.\end{aligned} \tag{22}$$

Again note the singularities in (20) and (22) at the cyclotron resonances (18).

By using the Newberger sum rule (NewBerger, 1982):

$$\sum_{n=-\infty}^{\infty} \frac{J_n^2(z)}{n + \chi} = \frac{\pi J_{\chi}(z) J_{-\chi}(z)}{\sin(\pi\chi)}, \tag{23}$$

in (20), we obtain

$$a = i\bar{\nu}_{\|} \frac{\pi J_{\chi}(k_{\perp} r_g \sin\theta) J_{-\chi}(k_{\perp} r_g \sin\theta)}{\sin(\pi\chi)}, \tag{24}$$

as an alternative, more compact expression for a, where

$$\chi = \frac{(\omega - k_{\|} v\mu + i\nu_{\perp})}{\Omega}, \tag{25}$$

is the normalized Doppler shifted frequency ω relative to the particle, taking into account perpendicular scattering.

Averaging (19) over the pitch angle cosine μ, yields a simple algebraic equation for $\tilde{F}_0$ with solution

$$\tilde{F}_0 = \frac{\langle \bar{Q}/[1 + (1 - \tau_{\|}/\tau_{\perp})a]\rangle}{1 - \langle a/[1 + (1 - \tau_{\|}/\tau_{\perp})a]\rangle}, \tag{26}$$

where the angular brackets in (26) denote an average over μ. For the case of isotropic scattering ($\tau_{\perp} = \tau_{\|}$) (26) simplifies to

$$\tilde{F}_0 = \frac{\langle \bar{Q}\rangle}{1 - \langle a\rangle}. \tag{27}$$

5 Long-scale, large-time asymptotics

From Fedorov et al. (1995), Kota (1994) and Webb et al. (2000), the long time asymptotics for F_0 can be obtained by investigating the dispersion equation

$$D(\mathbf{k}, s) = 1 - \langle a/[1 + (1 - \tau_{\|}/\tau_{\perp})a]\rangle = 0, \tag{28}$$

associated with the singular eigensolutions of (26). In particular, the diffusive behaviour of the solution follows from the large space-scale ($k \to 0$) and long time ($s \to 0$) behaviour of (28).

For example, consider the case of isotropic scattering ($\tau_{\|} = \tau_{\perp}$) for which $D(\mathbf{k}, s) = 1 - \langle a\rangle = 0$ is the singular manifold. Using the expansion of the Bessel functions in (24) for $|k_{\perp} r_g \sin\theta| << 1$, we obtain

$$a \approx i\frac{\bar{\nu}_{\|}}{\chi}\left(1 + \frac{(k_{\perp} r_g \sin\theta)^2}{2(\chi^2 - 1)} + O\left[(k_{\perp} r_g)^4\right]\right). \tag{29}$$

Using (29), we find

$$\langle a\rangle \approx -i\bar{\nu}_{\|}\delta\left(I_1 + \tfrac{1}{2}\left(k_{\perp}^2/k_{\|}^2\right) I_2\right), \tag{30}$$

for the approximate, pitch angle averaged value of a at long wavelengths, where

$$\begin{aligned}I_1 &= \frac{1}{2}\ln\left(\frac{\mu_c - 1}{\mu_c + 1}\right), \\ I_2 &= \frac{1}{4\delta^2}\Bigg\{[1 - (\mu_c - \delta)^2]\ln\left(\frac{\mu_c - \delta - 1}{\mu_c - \delta + 1}\right) \\ &+ [1 - (\mu_c + \delta)^2]\ln\left(\frac{\mu_c + \delta - 1}{\mu_c + \delta + 1}\right) \\ &- 2(1 - \mu_c^2)\ln\left(\frac{\mu_c - 1}{\mu_c + 1}\right)\Bigg\}, \\ \mu_c &= i\delta(\bar{s} + \bar{\nu}_{\perp}), \quad \delta = (k_{\|} r_g)^{-1}.\end{aligned} \tag{31}$$

From (30) and (31), the dispersion equation $D(\mathbf{k}, s) = 1 - \langle a\rangle = 0$, for $|s\tau| << 1$ and $kr_g << 1$ has the approximate solution

$$s = -\left(\kappa_{\|} k_{\|}^2 + \kappa_{\perp} k_{\perp}^2\right) + O\left(k^4\right), \tag{32}$$

where $\kappa_{\|}$ and $\kappa_{\perp}$ are the parallel and perpendicular diffusion coefficients in (9) for $\tau_{\|} = \tau_{\perp} = \tau$. Equation (32) is the dispersion equation for the diffusion equation obtained from (5), (8) and (9), but with no drift terms, since the background state is uniform.

On the other hand, if $k_{\perp}^2/k_{\|}^2 << 1$, the I_2 term can be dropped in (30), and the dispersion equation (28), has the approximate solution:

$$s \approx -\left(k_{\|}^2\kappa_{\|} + \tfrac{1}{5}k_{\|}^4\kappa_{\|}^2\tau\right) + O\left(k_{\|}^6\right). \tag{33}$$

The latter dispersion equation is equivalent to the equation:

$$\tfrac{1}{5}s^2\tau + s + k_{\|}^2\kappa_{\|} \approx 0. \tag{34}$$

In the space-time domain, (34) becomes the telegraph equation of Gombosi et al. (1993). Clearly, to obtain an equivalent telegraph equation including perpendicular diffusion, one needs to retain terms $O(k_{\perp}^4)$ in (29).

6 Pitch angle evolution and perpendicular diffusion

It is instructive to consider the integral equation (19) under the assumption that $|k_\perp r_g| << 1$, so that the approximation (29) for a applies. Equation (29) can be re-written in the form:

$$\left(\nu_\parallel/a - \nu_\perp\right)\tilde{f}_0 = \nu_\parallel(\tilde{F}_0 - \tilde{f}_0) + \nu_\parallel\tilde{Q}/a. \tag{35}$$

Using the usual Fourier space map

$$\frac{\partial}{\partial t} \to -i\omega \quad \text{and} \quad \nabla \to i\mathbf{k}, \tag{36}$$

and using the approximation (29) for a, (35) reduces to the approximate, integro-differential evolution equation

$$\begin{aligned}&\frac{\partial f_0}{\partial t} + v\mu\frac{\partial f_0}{\partial z}\\&-\frac{(\partial_t + v\mu\partial_z + \nu_\perp)}{(\partial_t + v\mu\partial_z + \nu_\perp)^2 + \Omega^2}\frac{v^2\sin^2\theta}{2}\nabla_\perp^2 f_0\\&= \nu_\parallel(F_0 - f_0) + \nu_\parallel\mathcal{F}^{-1}\left(\frac{\tilde{Q}}{a}\right),\end{aligned} \tag{37}$$

where $\nabla_\perp^2 = \partial_x^2 + \partial_y^2$ is the Laplacian operator transverse to the magnetic field, which is assumed to lie along the z-axis, and $\mathcal{F}^{-1}$ is the inverse Laplace and Fourier transform operator.

Assuming that f_0 evolves on much longer time scales than $\tau_\perp$, $\tau_\parallel$ and the gyro-period $2\pi/\Omega$ (i.e. $|f_{0t}/f_0| << \nu_\perp$, $\nu_\parallel$ and Ω) and on space scales much larger than the mean free paths $v\tau_\parallel$ and $v\tau_\perp$, then (37) can be approximated by the equation:

$$\begin{aligned}&\frac{\partial f_0}{\partial t} + v\mu\frac{\partial f_0}{\partial z} - \frac{v^2\nu_\perp(1-\mu^2)}{2(\nu_\perp^2 + \Omega^2)}\nabla_\perp^2 f_0\\&\quad= \nu_\parallel(F_0 - f_0) + \nu_\parallel\mathcal{F}^{-1}\left(\frac{\tilde{Q}}{a}\right),\end{aligned} \tag{38}$$

which is the pitch angle evolution equation for f_0 incorporating the effects of cross-field diffusion (the $\nabla_\perp^2 f_0$ term).

Multiplying (38) by $2\pi p^2$ and integrating (38) over μ from $\mu = -1$ to $\mu = 1$, using the diffusion approximation, and neglecting the source, or initial value term in (38) results in the usual diffusion equation (5) in the form:

$$\frac{\partial N}{\partial t} + \frac{\partial}{\partial z}\left(-\kappa_\parallel\frac{\partial N}{\partial z}\right) + \nabla_\perp\cdot(-\kappa_\perp\nabla_\perp N) = 0, \tag{39}$$

where $\kappa_\perp$ and $\kappa_\parallel$ are given by (9). In the derivation of (39) it is also necessary to take the first moment of (38) (i.e. multiply (38) by $2\pi p^2 v\mu$ and integrate over μ from $\mu = -1$ to $\mu = 1$, and use the diffusion approximation to find the diffusive streaming parallel to the field). It is clear that accurate approximate solutions of (38) can be obtained by expanding the distribution function in terms of Legendre polynomials, and taking moments of (38) (e.g., Gombosi et al. 1993; Lu et al., 2001).

7 Concluding Remarks

From the explicit solution for $\tilde{F}_0$ in (26), the complete solution for $f(\mathbf{r}, \mathbf{p}, t)$ for the case of Dirac-delta initial data in position, pitch angle and gyro-phase, can be constructed by Laplace-Fourier inversion, by first determining $\tilde{F}_0$ from (26), and using the result to determine $\tilde{f}_0$ from (19), and then obtain $\tilde{f}$ from (13), followed by Laplace and Fourier inversion to determine f. A multiple scattering analysis (e.g. Webb et al. 2000); eigenfunction/moment equation methods should reveal further aspects of the solution.

There are several outstanding issues raised by the above analysis. For example, in a non-uniform background magnetic field, there is a non-zero contribution to the divergence of the particle current due to curvature and gradient drifts associated with the antisymmetric diffusion coefficient κ_A. It is of interest to determine whether the effects of these drifts can be included in a pitch angle evolution equation analogous to (38) in this case. It is also of interest to investigate higher order transport effects in the model, e.g. the incorporation of cosmic ray inertial effects in telegraph type equations for cosmic ray transport including cross-field diffusion, that generalize the telegraph equation obtained by Gombosi et al. (1993). Other aspects of cosmic ray transport theory that are raised by the analysis, concern the form of the pitch angle evolution equation obtained by Skilling (1975) for particle transport in the solar wind, or its relativistic generalization (e.g. Webb, 1985) when cross field transport is included, and the role of cross field transport effects on cosmic ray viscosity, and non-inertial acceleration effects.

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