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The relativistic box model

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Abstract. The equations governing the 'box model' approach to diffusive shock acceleration are derived and generalised to apply to acceleration at relativistic shocks. The difficulties encountered in describing the particle spectrum close to the point where losses balance gains are discussed in relation to the question of 'pile-ups'.

1 Introduction

The theory of diffusive particle acceleration at shock fronts involves, in the simplest, test-particle approximation the solution of the cosmic ray transport equation in the variables $\boldsymbol{x}$ (position), p (magnitude of the momentum) and time t. A formal expression for the Green's function is known for the case of a plane shock (Webb et al , 1995). However, the inclusion of synchrotron losses by the accelerated particles significantly complicates the problem: an analytic solution has been found only for the stationary case and for constant (energy independent) diffusion coefficient (Webb et al , 1984; Heavens & Meisenheimer , 1987); Time-dependent solutions, or solutions with a diffusion coefficient which depends on energy require a numerical approach.

These are of limited usefulness for detailed modelling of the spectra of astrophysical sources of synchrotron radiation such as supernova remnants and blazars, since one requires a solution that can be computed rapidly. This has led to the development of numerous models in which the space dependence of the problem is approximated as homogeneous within one or two zones [see Kirk, Melrose & Priest (1994)]. Two-zone models, with one acceleration zone and one cooling zone, are the most useful. The equations governing the acceleration zone are equivalent to those of the 'box' models, which have recently been generalised to include the effects of energy dependent diffusion Drury et al (1999). In this paper, the box model equations are first re-derived in a manner which clarifies their range of validity. They are then generalised further to the case of relativistic shocks, for which no closed-form analytic solutions of the transport equation are available.

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2 Nonrelativistic boxes

For nonrelativistic flows the equation governing the particle density $n(p,x,t)$ in a prescribed, one-dimensional velocity field $u(x)$ can be written in conservation form:

$$\frac{\partial n}{\partial t}+\frac{\partial \Phi}{\partial x}+\frac{\partial \Psi}{\partial p} \quad = \quad 0 \tag{1}$$

Here the fluxes Φ and Ψ in configuration space and in momentum space, respectively, are defined as

$$\Phi \quad = \quad [u-\kappa\partial/\partial x]n \tag{2}$$

$$\Psi \quad = \quad [-(p/3)(\mathrm{d}u/\mathrm{d}x)-\alpha p^2]n \tag{3}$$

κ is the diffusion coefficient, which may be a function of momentum p, and the parameter $\alpha=(4\sigma_{\mathrm{T}}/3m^2c^2)(B^2/8\pi)$, determines the synchrotron loss rate of an individual particle, with σ_{T} the Thomson cross-section and B the magnetic field.

Consider a simple discontinuous velocity profile $u=u_-$ for $x<0$, and $u=u_+$ for $x>0$, with $u_->u_+>0$. To find the box equations, one can proceed by defining spatial boundaries at $x=x_-$ and $x=x_+$ and integrating Eq. (1) between the two. If x_- lies sufficiently far upstream ($x_-\rightarrow -\infty$) the flux $\Phi(x_-)$ vanishes. At the other boundary, x_+, this is not the case, but the essential simplification inherent in the box model results if it is permissible to neglect the diffusive part of the spatial flux there, i.e.,

$$\Phi(x_+)=u_+n(x_+) \tag{4}$$

One then finds

$$\frac{\partial}{\partial t}(n_+L)+\frac{\partial}{\partial p}\left[\left(\frac{\Delta u}{3}\xi p-\alpha_+p^2L_{\mathrm{s}}\right)n_+\right]+u_+n_+ \quad = \quad 0 \tag{5}$$

where $n_+(p,t) = n(p,x_+,t)$, $\alpha_+ = \alpha(x_+)$ and $\xi(p,t) = n(p,0,t)/n_+$, and two measures of the box size have been introduced:

$$L(p,t) \quad = \quad \int_{-\infty}^{x_+} \mathrm{d}x' n(p,x',t)/n_+ \tag{6}$$

and

$$L_\mathrm{s}(p,t) \quad \equiv \quad \int_{-\infty}^{x_+} \mathrm{d}x' \alpha(x') n(p,x',t)/(\alpha_+ n_+) \tag{7}$$

The first of these, $L(p)$, which is of the order of κ/u_- in the absence of losses, is related to the momentum dependent box width introduced by Drury et al (1999). The second merely takes into account the possibility that synchrotron losses are influenced by the compression of the magnetic field; for a parallel shock $L \equiv L_\mathrm{s}$. In this case, the nonrelativistic box model equations [eq. (13) of Drury et al (1999)] are obtained, provided the downstream boundary is placed at a point where the density is approximately equal to that at the shock front, in which case $\xi = 1$.

Rewriting in terms of the number of particles in the box $N(p,t) = Ln_+$: one finds

$$\frac{\partial N}{\partial t} + \frac{\partial}{\partial p}\left[\frac{p}{t_\mathrm{acc}} - \alpha_+ p^2 \frac{L_\mathrm{s}}{L}\right] N + \frac{N}{t_\mathrm{esc}} \quad = \quad 0 \tag{8}$$

where the acceleration time $t_\mathrm{acc} = 3L/(\Delta u \xi)$ and escape time $t_\mathrm{esc} = L/u_+$ follow from a microscopic consideration of the acceleration process, or, alternatively, the loss-free Green's function. When combined with an equation for the particle density $n_\mathrm{c}(p,x,t)$ in the cooling zone,

$$\frac{\partial n_\mathrm{c}}{\partial t} + u_+ \frac{\partial n_\mathrm{c}}{\partial x} - \frac{\partial}{\partial p}(\alpha_+ p^2 n_\mathrm{c}) \quad = \quad \frac{N}{t_\mathrm{esc}}\,, \tag{9}$$

this system has been widely used to model synchrotron spectra [e.g., Kirk et al (1998)]; a simple time-dependent analytic solution is available for (almost) arbitrary energy dependence of t_acc and t_esc.

The approximations used in the above derivation ($\Phi(x_+) = u_+ n_+$ and $\xi = 1$) are acceptable for low momenta, where the downstream distribution remains close to the loss-free solution ($n(p,x,t) = \mathrm{constant}$ for $x > 0$) and the diffusive flux is small. However, they run into difficulty near the cut-off momentum $p_\mathrm{max} = \Delta u/(\alpha_+ L)$: in order to keep $\xi \approx 1$, it is necessary to place the downstream boundary so close to the shock front that particles with $p = p_\mathrm{max}$ do not cool appreciably between leaving the shock and crossing the boundary. This implies $x_+ < u_+/(\alpha_+ p)$. However, at this position, losses impose a gradient on n_+, that leads to a diffusive flux of the order of

$$\kappa \frac{n_+}{u_+/\alpha_+ p} \approx \left(\frac{\kappa \Delta u}{u_+^2 L}\right)\left(\frac{p}{p_\mathrm{max}}\right) u_+ n_+ \tag{10}$$

This flux is neglected in the model, but is about the same magnitude as the advective flux for $p \approx p_\mathrm{max}$. Thus, box-models are useful for modelling the spectrum of accelerated particles at $p \ll p_\mathrm{max}$ but are not appropriate tools for the investigation of possible pile-ups close to p_max [e.g., Schlickeiser (1984); Protheroe & Stanev (1999)].

3 Relativistic boxes

If the bulk plasma speed is relativistic, one must start from the equation governing the angular dependent distribution function f, representing the density of particles in phase space [Eq. (10) of Kirk et al (1988)]. For a stationary velocity profile, and assuming ultra-relativistic particles and cylindrical symmetry of the distribution about the direction of the shock normal, this equation reads:

$$\begin{aligned} &\Gamma(1+u\mu)\frac{\partial f}{\partial t} + \Gamma(u+\mu)\frac{\partial f}{\partial x} \\ &-\Gamma(u+\mu)\frac{\mathrm{d}u}{\mathrm{d}x}\Gamma^2\left(\mu p \frac{\partial f}{\partial p} + (1-\mu^2)\frac{\partial f}{\partial \mu}\right) \\ &\quad -\frac{1}{p^2}\frac{\partial}{\partial p}\left(\alpha p^4 f\right) \;=\; \frac{\partial}{\partial \mu} D_{\mu\mu} \frac{\partial f}{\partial \mu} \end{aligned} \tag{11}$$

where $\Gamma = (1-u^2)^{-1/2}$, u is the fluid speed, in units of the speed of light, α is as defined in the non-relativistic case and $D_{\mu\mu}$ is the pitch-angle diffusion coefficient. Other transport coefficients describing, for example, second order Fermi acceleration, are usually less important and have been omitted in (11). Position, x, and time, t, are measured in the lab. frame, but the momentum p and (cosine of) pitch angle μ are measured in the local rest frame of the fluid. An integration over angles leads to:

$$\begin{aligned} &\frac{\partial}{\partial t}\left(\Gamma J + \Gamma u H\right) + \frac{\partial}{\partial x}\left(\Gamma u J + \Gamma H\right) \\ &\quad + \frac{1}{p^2}\frac{\partial}{\partial p}\left[-p^3(K+uH)\frac{\mathrm{d}}{\mathrm{d}x}(\Gamma u) - \alpha p^4 J\right] \;=\; 0 \end{aligned} \tag{12}$$

The moments of the phase space-density f with respect to the cosine μ of the angle to the shock normal which appear in this equation are analogous to those defined in the theory of radiative transfer:

$$(J,H,K) \quad = \quad 2\pi \int_{-1}^{+1} \mathrm{d}\mu f(p,\mu,x,t)(1,\mu,\mu^2) \tag{13}$$

Proceeding as in the nonrelativistic case, one now places a downstream boundary, x_+, at a point where the distribution has relaxed to isotropy, i.e., where $H = 0$, $J = 3K$. Then, integrating from $-\infty$ to x_+:

$$\begin{aligned} &\frac{\partial}{\partial t}(L\Gamma_+ p^2 J_+) + \frac{\partial}{\partial p}\left\{\left[\frac{\Delta(\Gamma u)}{3\Gamma_+}\xi p - \frac{\alpha_+ p^2 L_\mathrm{s}}{\Gamma_+}\right]\Gamma_+ p^2 J_+\right\} \\ &\quad + u_+ \Gamma_+ p^2 J_+ \;=\; 0 \end{aligned} \tag{14}$$

where $\Delta(\Gamma u) = (\Gamma_- u_- - \Gamma_+ u_+)$, $\xi = \langle 3(K+uH)\rangle / J_+$ (with $\langle \ldots \rangle$ the arithmetic mean across the discontinuity) and the effective lengths are now defined via

$$L \quad = \quad \int_{-\infty}^{x_+} \mathrm{d}x' (J + uH)/J_+ \tag{15}$$

$$L_\mathrm{s} \quad = \quad \int_{-\infty}^{x_+} \mathrm{d}x' \alpha J/J_+ \tag{16}$$

In terms of $N = L\Gamma_+ p^2 J_+$, Eq. (8) is recovered, with the replacement $\alpha_+ \Rightarrow \alpha_+/\Gamma_+$, $t_\mathrm{acc} = 3L\Gamma_+/[\Delta(\Gamma u)\xi]$ and $t_\mathrm{esc} = L/u_+$.

However, in contrast to the nonrelativistic case, the approximation $\xi \approx 1$ is not valid even if losses are neglected. This is because the relativistic fluid motion imposes an anisotropy on the particle distribution at the shock, leading to a particle flux there which cannot be approximated as the advective flux. Within the box picture it is not possible to determine ξ. However, $t_{\rm acc}$ and $t_{\rm esc}$ are related quite generally to the power-law index s of the stationary distribution produced by a shock front: $t_{\rm acc} = (s-3)t_{\rm esc}$, so that

$$\xi = \frac{3\Gamma_+ u_+}{(s-3)\Delta(\Gamma u)} \quad (17)$$

The appropriate value of s can be determined for a given relativistic shock front using, for example, the method of Kirk et al (2000). For the purpose of spectral modelling, however, it is usually found directly from the low frequency spectral index of synchrotron radiation.

4 Conclusions

The approach to particle acceleration at shocks that involves treating the distribution as homogeneous over certain spatial regions or 'boxes' is a valuable tool for modelling source spectra. This is mainly because it is possible to include processes such as losses and time-dependent injection and still obtain easy to compute analytic solutions. The equations contain acceleration and escape rates which must be found from a more detailed theory. In the relativistic case, the same equations can be derived, with minor reinterpretation of the expressions for the rates. The stationary, low momentum spectrum is, however, not related to the compression ratio of the shock by a simple algebraic expression. Instead, this quantity must be used as an input parameter in modelling, or, alternatively, computed using a semi-analytic method (Kirk et al , 2000).

However, a drawback of this approach, that applies to both the nonrelativistic and relativistic cases, is that the distribution function is not described accurately close to the point where losses balance gains. This implies that the controversial question of whether 'pile-ups' can be expected near the maximum energy cannot be addressed using box models. Physically, the reason is that pointed out by Drury et al (1999): box models assume the escape probability from the acceleration zone is the same for all particles, whereas, in reality, a range of escape probabilities occurs, depending on the instantaneous particle position. This range of escape rates determines the shape of the distribution at the cut-off.

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References

Drury L.O'C., Duffy P., Eichler D., Mastichiadis A. 1999 A&A 347, 370
Heavens A.F., Meisenheimer K. 1987 MNRAS 225, 335
Kirk J.G., Guthmann A.W., Gallant Y.A., Achterberg A. 2000 ApJ 542, 235
Kirk J.G., Melrose D.B., Priest E.R. 'Plasma Astrophysics' (Springer-Verlag, Heidelberg)
Kirk J.G., Rieger F.M. Mastichiadis A. 1998 A&A 333, 452
Kirk J.G., Schlickeiser R., Schneider P. 1988 ApJ 328, 269
Protheroe R.J. & Stanev T. 1999 Astroparticle Phys. 10, 185
Schlickeiser R. 1984 A&A 136, 227
Webb G.M., Drury L.O'C., Biermann P.L. 1984 A&A 137, 185
Webb G.M., Zank G.P., Ko C.M., Donohue D.J. 1995, ApJ 453, 178