

Dispersions of composite parameters for Molière angular distribution due to difference of particles and scatterers

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Abstract. For relativistic electrons of fixed energy, depth variation of the two parameters B and θ_M characterizing the Molière angular distribution has been represented by a universal function described in the composite variables. Under the moderate relativistic conditions with ionization, small dispersions from the universal function arise due to the configuration of Molière screening model. We have investigated the magnitude of dispersion on the figure plotted in the composite variables, for various energies of charged particles and for various substances to traverse through with ionization. We have found dispersion of the depth-variation curve arising from the difference of scattering substances from light to heavy ones is negligibly small on the figure, although the dispersion due to the difference of rest-mass has been found not negligible. The results will bring us simple and rapid derivations of Molière angular distribution for charged particles traversing through pure substances.

1 Introduction

Although various types of theory have been proposed to predict the multiple Coulomb scattering process, Molière theory (Molière, 1947, 1948; Bethe, 1953) still keeps the highest quality among all. The solution is highly accurate reflecting single, double, and plural scatterings other than multiple scattering; the distribution is described in series expansion of rapid convergence (Bielajew, 1994), by the expansion parameter B of order of ten; the series expansion is composed of universal functions of plain expressions with two characteristic parameters. Moreover the Molière theory is further improved to take account ionization loss (Nakatsuka, 1999a), by using the Kamata-Nishimura formulation of the theory (Kamata and Nishimura, 1958; Nishimura, 1967).

Reconstruction of the Molière theory by Kamata-Nishimura formulation is improving the traditional theory by Molière and Bethe in contents, applications, and understandings of

the theory. One of the most superior aspects of the Kamata-Nishimura formulation is that we can describe the Molière theory in far simple way by introducing the constants Ω and K specific to substance (Kamata and Nishimura, 1958; Nishimura, 1967; Nakatsuka, 2001). It will be valuable for our simulation works (Messel and Crawford, 1970; Nelson et al., 1985; Heck et al., 1998) where the theory is referred to vast times in tracing charged particles and for our designings and analyses of experiments where rapid and frequent derivations of the distribution are required (Yamashita et al., 1996).

The Molière angular distribution is characterized by two parameters, the expansion parameter B and the unit of Molière angle θ_M . In case of relativistic electrons with fixed energies, B and θ_M of Molière angular distribution are described by composite variables in universal functions irrespective of substances. Under the moderate relativistic conditions with ionization, there still remains a term in the characteristic parameters which depends on substances explicitly even after we introduced the above composite variables, in case we adopt the Molière screening model (Molière, 1947, 1948; Bethe, 1953). If we assume the Born-type screening angle where the characteristic screening angle χ_a is proportional to the Born screening angle χ_0 (Scott, 1963) or under the Molière screening angle with the small enough Born parameter, $zZ/(137\beta) \ll 1$, it satisfies $\beta' \simeq \beta$. In this case, dispersions of the characteristic parameters from the universal function described in the composite variable disappear.

Satisfaction of $\beta' \simeq \beta$, so that the acceptance of the universal function, is important for rapid and plain derivations of Molière angular distribution of charged particles traversing through pure substances. It will be further important for derivations of the distribution in mixed or compound substances, the stochastic mean among the composing substances becomes far easy to obtain in this situation (Nakatsuka, 2001). So we have investigated the feasibility to accept the universal function on the figure of the characteristic parameters expressed by the composite variables, for various substances around us from light to heavy substances.

2 Characteristic Parameters B and θ_M of Molière Angular Distribution

The Molière angular distribution $f(\vartheta)2\pi\vartheta d\vartheta$ is represented by the series

$$f(\vartheta) = f^{(0)}(\vartheta) + B^{-1}f^{(1)}(\vartheta) + B^{-2}f^{(2)}(\vartheta) + \dots, \quad (1)$$

where the Molière angles ϑ is defined by

$$\vartheta = \theta/\theta_M. \quad (2)$$

The functions $f^{(k)}$ are the universal functions defined in Molière (1947, 1948), except the factor of 2π .

We find the Molière angular distributions are characterized by two parameters, the expansion parameter B and the unit of Molière angle θ_M . So we want to discuss the dispersions of Molière angular distribution due to various conditions, by these parameters.

3 Molière Theory Described in Kamata-Nishimura Formulation

According to the Kamata-Nishimura formulation of Molière theory (Kamata and Nishimura, 1958; Nishimura, 1967), the diffusion equation of the angular distribution for charged particles of charge z , rest-mass mc^2 , and velocity β , traversing through substance of atomic number Z with ionization, is represented by

$$\frac{\partial \tilde{f}}{z^2 \partial t} = -\frac{\zeta^2}{w^2} \tilde{f} \left\{ 1 - \frac{1}{\Omega} \ln \frac{\beta'^2 \zeta^2}{w^2} \right\} + \varepsilon \frac{\partial \tilde{f}}{\partial E}, \quad (3)$$

in the Fourier space, where the traversed thickness t is measured in the radiation length (Particle Data Group, 2000). We have defined

$$w = 2pv/K = \frac{2E}{K} \left\{ 1 - \left(\frac{mc^2}{E} \right)^2 \right\} \quad (4)$$

and

$$\beta'^2 = \frac{1.13 + 3.76\alpha^2}{1.13 + 3.76\alpha_0^2} \beta^2, \quad (5)$$

with

$$\alpha = \frac{zZ}{137\beta} \quad \text{and} \quad \alpha_0 = \frac{Z}{137}. \quad (6)$$

K and Ω denote Kamata-Nishimura constants specific to the substance (Kamata and Nishimura, 1958; Nishimura, 1967; Nakatsuka, 2001). We assume the ionization loss of a constant rate, dissipating $z^2\varepsilon$ in unit radiation length, so that we have

$$E = E_0 - z^2\varepsilon t. \quad (7)$$

The solution of Eq. (3) can be expressed as

$$\tilde{f} = \frac{1}{2\pi} \exp \left\{ -\frac{\theta_M^2 \zeta^2}{4} \left(1 - \frac{1}{\Omega} \ln \frac{\theta_M^2 \zeta^2}{4\nu z^2 t / \beta'^2} \right) \right\}, \quad (8)$$

where θ_G denotes the gaussian root-mean-square angle taking account rest mass (Nakatsuka, 1999a), derived from

$$\begin{aligned} \theta_G^2 &= \int_0^t \frac{4z^2}{w^2} dt \\ &= \frac{K^2}{2\varepsilon mc^2} \left\{ \frac{mc^2}{pv} - \frac{mc^2}{p_0 v_0} + \frac{1}{2} \ln \frac{(E_0 - mc^2)/(E - mc^2)}{(E_0 + mc^2)/(E + mc^2)} \right\}, \end{aligned} \quad (9)$$

and the scale factor ν is determined from

$$\ln \frac{\nu}{\beta'^2} = \ln \frac{\theta_G^2}{4z^2 t} - \frac{4z^2}{\theta_G^2} \int_0^t \frac{1}{w^2} \ln \frac{\beta'^2}{w^2} dt. \quad (10)$$

Applying the translation formula indicated in Nakatsuka (1999b), the solution (8) is reduced to the Molière form,

$$\tilde{f} = \frac{1}{2\pi} \exp \left\{ -\frac{\theta_M^2 \zeta^2}{4} \left(1 - \frac{1}{B} \ln \frac{\theta_M^2 \zeta^2}{4} \right) \right\}, \quad (11)$$

with the expansion parameter B and the unit of Molière angle θ_M :

$$B - \ln B = \Omega - \ln \Omega + \ln(\nu z^2 t / \beta'^2), \quad (12)$$

$$\theta_M = \theta_G \sqrt{B/\Omega}. \quad (13)$$

Thus we get the Molière angular distributions indicated in the previous section.

4 Characteristic Parameters Under The Extreme Relativistic Condition and Introduction of Composite Variables

For singly charged particles with extreme relativistic energies, it satisfies

$$E \gg mc^2. \quad (14)$$

Then we have

$$w \simeq 2E/K, \quad (15)$$

$$\beta \simeq 1 \quad (16)$$

from Eq. (4), so that the two parameters B and θ_M are determined by

$$B - \ln B = \Omega - \ln \Omega + \ln \nu z^2 t, \quad (17)$$

$$\theta_M = \theta_G \sqrt{B/\Omega}, \quad (18)$$

where

$$\nu = e^2 (E/E_0)^{(E_0+E)/(E_0-E)}. \quad (19)$$

and

$$\theta_G^2 = \frac{K^2 z^2 t}{E_0 E}. \quad (20)$$

As it holds

$$B - \ln B = \ln \left(\nu \frac{t}{\Omega e^{-\Omega}} \right), \quad (21)$$

$$\theta_M^2 / \frac{K^2 e^{-\Omega}}{E_0^2} = \frac{B}{E/E_0} \frac{t}{\Omega e^{-\Omega}}, \quad (22)$$

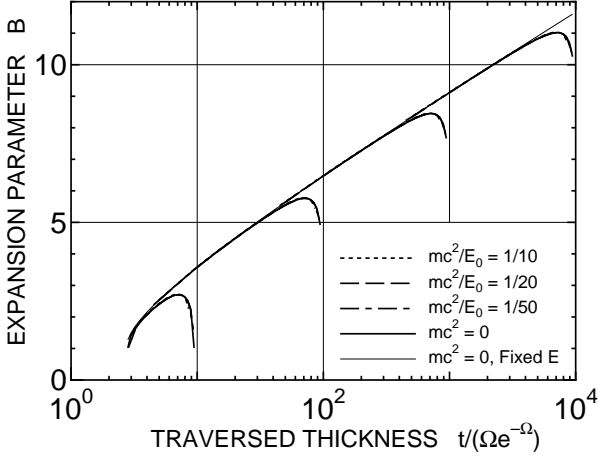


Fig. 1. Discrepancies of B due to the different rest masses. Incident energies E_0/ε correspond to $10, 10^2, 10^3,$ and 10^4 in unit of $\Omega e^{-\Omega}$, from left to right.

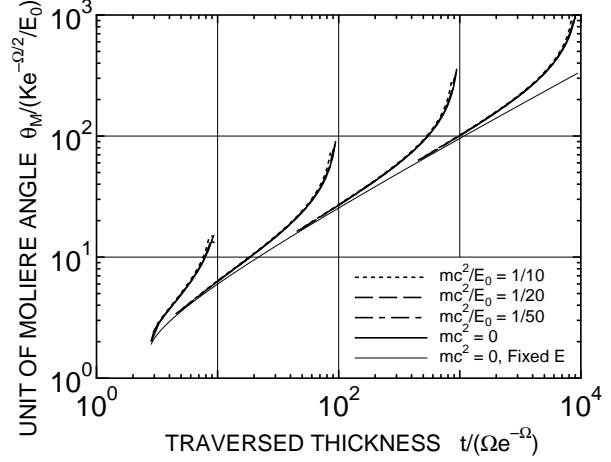


Fig. 2. Discrepancies of θ_M due to the different rest masses. Incident energies E_0/ε correspond to $10, 10^2, 10^3,$ and 10^4 in unit of $\Omega e^{-\Omega}$, from left to right.

for singly charged particles, we find the characteristic parameters are described universally irrespective of substances, by using the composite variables $t/(\Omega e^{-\Omega})$ and $\theta_M/(K e^{-\Omega/2}/E_0)$. The unit $\Omega e^{-\Omega}$ for the traversed thickness is almost the same as the mean free path of the single scattering larger than the screening angle, measured in the radiation length. It should be noted that the characteristic parameters represented in the composite variable, B and $\theta_M/(K e^{-\Omega/2}/E_0)$, are functions of fractional energy, in case of the extreme relativistic condition.

5 Discrepancy of Molière Angular Distribution Arising from The Difference of Rest Mass

We investigate dispersions of the characteristic parameters, B and θ_M , due to the difference of rest-mass mc^2 , for singly charged particles with moderate relativistic energies. We assume the Born parameter be small enough, $zZ/137\beta \ll 1$, which is realized at e.g. the penetration through light substances. Then it satisfies $\beta' \simeq \beta$, and we can determine the characteristic parameters as

$$B - \ln B = \Omega - \ln \Omega + \ln(\nu t/\beta^2), \quad (23)$$

$$\theta_M = \theta_G \sqrt{B/\Omega}, \quad (24)$$

with θ_G from Eq. (9), and ν is derived from

$$\ln \frac{\nu}{\beta^2} = \ln \frac{\theta_G^2}{4z^2 t} - \frac{4z^2}{\theta_G^2} \int_0^t \frac{1}{w^2} \ln \frac{\beta^2}{w^2} dt. \quad (25)$$

The scale factor ν , so that B and θ_M , are functions of E_0/mc^2 and E/mc^2 in this case.

We compare the results of B and θ_M for various E_0/mc^2 of 10, 20, 50, and ∞ , in Figs. 1 and 2. A slight differences appear with increase of the fractional thickness $t/(E_0/\varepsilon)$ especially for curves of lower values of E_0/mc^2 .

6 Dispersion of Molière Angular Distribution Arising From The Molière Screening Angle

Under the extreme relativistic condition, we could represent the characteristic parameters B and θ_M from Eqs. (17), (18) universally irrespective of substances, by describing the traversed thickness and the unit of Molière angle in composite variables, $t/(\Omega e^{-\Omega})$ and $\theta_M/(K e^{-\Omega/2}/E_0)$ respectively.

Under the Molière screening model with moderate relativistic energies, the characteristic parameters B and θ_M derived from Eqs. (12), (13) still require the explicit Z in the term β' even if we use the above composite variables. Difference of β' from β arises from energy dependence of the ratio, Molière screening angle to Born screening angle. In this case, we cannot describe the characteristic parameters universally by the composite variables, in the definite sense. But in case it satisfies $\beta' \simeq \beta$, which is realized in case of Born parameter to be small enough, B and θ_M could be determined from Eqs. (23), (24), and be described in universal expressions by the composite variables.

We examine whether the relation $\beta' \simeq \beta$ satisfies or not, so that the universal relations satisfy or not, on the practical substances around us. The B and θ_M derived from β' by Eqs. (12), (13) and those from β by Eqs. (23), (24) are compared on substances C, Fe, and Pb in Figs. 3 and 4. We cannot find any visible differences more than 1 percent between them within passage of energy loss less than 80 percent.

7 Conclusions and Discussions

We have investigated the dispersions of characteristic parameters B and θ_M due to the differences of rest mass and substance, on the figures plotted in composite variables $t/(\Omega e^{-\Omega})$ and $\theta_M/(K e^{-\Omega/2}/E_0)$.

Dispersions of the characteristic parameters B and θ_M due to the difference of rest mass are found not negligible among

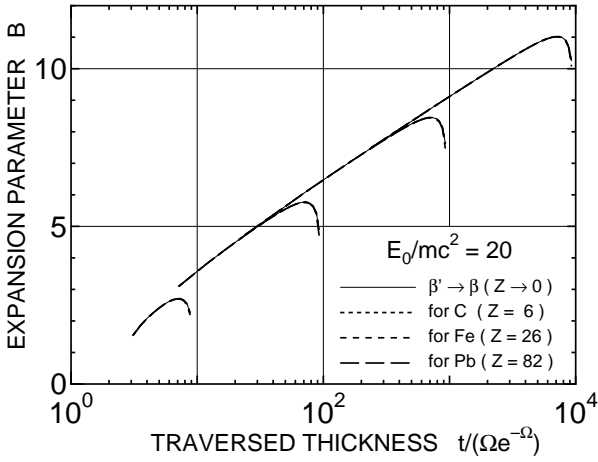


Fig. 3. Discrepancies of B due to the different Molière screening angles from Born ones by substance. Incident energies E_0/ε correspond to $10, 10^2, 10^3,$ and 10^4 in unit of $\Omega e^{-\Omega}$, from left to right.

E_0/mc^2 of 10, 20, 50, and ∞ as shown in Figs. 1 and 2. Instead, dispersions due to the difference of substance are found negligibly small among C, Fe, Pb, and a substance of lowest Z or of negligible Born parameter, for wide ranges of energy and long passages of traverse. It means we can replace β' simply by β in Eq. (12) for derivation of the expansion parameter B . So we can decide the characteristic parameters B and θ_M universally irrespective of substances around us by using the composite variables, as shown in Figs. 3 and 4.

Satisfaction of $\beta' \simeq \beta$ will make the practical derivation of Molière angular distribution far simple for pure substances. This fact will also be essential in practical derivations of the distribution for mixed or compound substances (Nakatsuka and Nishimura, 2001).

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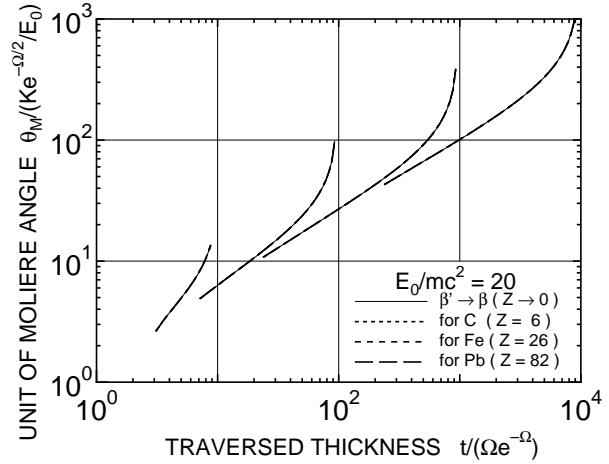


Fig. 4. Discrepancies of θ_M due to the different Molière screening angles from Born ones by substance. Incident energies E_0/ε correspond to $10, 10^2, 10^3,$ and 10^4 in unit of $\Omega e^{-\Omega}$, from left to right.

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