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The small pitch angle scattering

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Abstract. The diffusive particle propagation and its pitch angle scattering is studied using kinetic equation of the Fokker - Planck form. Due to existence of the strong regular magnetic field (MF) the particles are preferable propagated along the mean MF direction and undergo the pitch angle scattering with respect to it. The paper deals with solution of the equation for particle distribution function in the second order approximation in the pitch angle. The exact analytical solution is obtained in an integral form. The well known solution in the first order pitch angle approximation can be restored performing the small time limit in the result. Unlike the first order solution the obtained solution in the second approximation rightly shows that the pitch angle diffusion is closely connected with the particle transport along the mean MF. The expression for particle density for the point instantaneous unidirectional source also has been obtained.

1 Introduction

Study of multiple charged particle scattering in magnetic field with random inhomogeneities as scattering centers is important in turbulent theory plasma (Shkarofski et.al., 1966), in problems of cosmic ray particle propagation through cosmic media (Jokipii, 1966; Dorman and Kats, 1977), and many other problems of particle transport (Case and Zweifel, 1967). If the magnetic field is sufficiently strong that the Larmor radius of particle $R_L \ll \lambda$ (λ - the particle mean free path with respect to its scattering in magnetic field inhomogeneities), the averaging over particle spiral motion around the magnetic field can be performed, and one can restrict himself to a simple rectilinear system.

The diffusive particle propagation and its pitch angle scattering along the mean magnetic field is governed by kinetic equation of the Fokker - Planck form, and the particle distribution function, $\widetilde{f}$, depends only on location, x, the pitch angle, θ, and time, t. (In the following the tilde denotes the function of dimension variables.) Therefore, the kinetic equation reduces to (Gleeson and Axford, 1967; Galperin et.al., 1971):

$$\begin{aligned} & \partial_t \widetilde{f} + v\cos\theta\, \partial_x \widetilde{f} \\ = \quad & \frac{v}{\lambda \sin\theta}\, \partial_\theta \sin\theta\, \partial_\theta \widetilde{f} + \delta(x)\,\delta(t)\, \frac{\delta(\theta-\theta_0)}{\sin\theta_0}\,. \end{aligned} \tag{1}$$

Here v is the particle velocity. Introducing the dimensionless variables, $y = x/\lambda$, $\tau = vt/\lambda$, and $\mu = \cos\theta$, one obtain the equivalent equation for $f(y,\tau,\mu)$:

$$\partial_\tau f + \mu\, \partial_y f = \partial_\mu f (1-\mu^2)\, \partial_\mu f + \frac{1}{\lambda}\delta(y)\,\delta(\tau)\,\delta(\mu-\mu_0)\,. \tag{2}$$

Note that cross-field transport (i.e. perpendicular diffusion and drift, energy change, or adiabatic focusing) is not included into the model.

The paper deals only with an exact solution of this equation. The obtaining of analytical solution of this type equation without any approximation is problematic and its simplification depends on concrete model. The approximation of large pitch angle (i.e. $\mu \ll 1$) has been studied by Earl (1996). Here the model is considered, when particle pitch angle, θ, is small, i.e. $\theta \ll 1$.

2 The first order pitch angle approximation

Using the Fourier transform in the space variable, y, and the Laplace transform in the time, τ, the Eq. (2) gives the ordinary differential equation

$$\frac{d}{d\mu}(1-\mu^2)\frac{d}{d\mu}\, f - \mathrm{i}k\mu\, f - s\, f = -\frac{1}{\lambda}\delta(\mu-\mu_0)\,. \tag{3}$$

The equation is similar to known one generating the Coulomb spheroidal functions (see on p. 146 of Komarov et.al., 1976), which reads

$$\begin{aligned} & \frac{d}{d\mu}(1-\mu^2)\frac{d}{d\mu}\, u(\mu) \\ + \quad & \left[-p^2(1-\mu^2) - \frac{m^2}{1-\mu^2} + b\,\mu + \lambda\right] u(\mu) = 0\,. \end{aligned} \tag{4}$$

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Here $m = 0, 1, 2, ...;\ \ 0 \le p < \infty$, and, b, λ are parameters. In our case, $m = p = 0,\ b = -\mathrm{i}k,\ \lambda = s$. In this limit case, solutions of (5) does not lead to known special functions (Komarov et.al., 1976, pp.141-146), therefore, some approximation of the Eq. (2) is necessary.

The simplest approximation correspond to very small pitch angle, θ, when one can put $\sin\theta \to \theta$ and $\cos\theta \to 1$. In other words, the equality $\sin^2\theta + \cos^2\theta = 1$ is fulfilled up only to the first order in θ, i.e. $\theta^2 \to 0$. In this case the function $f_1(y, \tau, \theta)$ in the first order approximation,

$$\partial_\tau f_1 + \partial_y f_1 = \frac{1}{\theta}\,\partial_\theta\,\theta\,\partial_\theta f_1 + \frac{1}{\lambda}\delta(y)\,\delta(\tau)\frac{\delta(\theta-\theta_0)}{\theta_0}\,. \qquad (5)$$

Analogous stationary equation for the small pitch angle including the focusing term has been investigated by Galperin et. al. (1971) for the case of particle injection along the magnetic field, $\theta_0 = 0$. Then Dorman and Kats (1974) have obtained the solution of the stationary equation for the small non-zero $\theta_0 \ll 1$. The non-stationary case has been studied by the same authors (Dorman and Kats, 1977) in the first approximation of small θ and θ_0. The solution $f_1(y, \tau, \theta, \theta_0)$ of (5) in the first approximation is

$$f_1 = \frac{1}{2\lambda\tau}\ \exp\left[-\frac{\theta^2+\theta_0^2}{4\tau}\right]\ I_0\left(\frac{\theta\theta_0}{2\tau}\right)\ \delta(y-\tau)\,, \qquad (6)$$

where $I_0(x)$ is the zeroth order Bessel function of imaginary argument (the hyperbolic Bessel function). The last equation rewritten into the variables $\{x, t\}$ reads

$$\widetilde{f}_1 = \frac{\lambda}{2vt}\ \exp\left[-\lambda\frac{\theta^2+\theta_0^2}{4vt}\right]\ I_0\left(\lambda\frac{\theta\theta_0}{2vt}\right)\ \delta(x-vt)\,. \qquad (7)$$

The δ-function in Eq. (6) expresses the particle free propagation along the axis y. All particles are located in the plane of $y = \tau$, and, the angular distribution is rather wide already at a small time past the particle injection (see in Fig. 1).

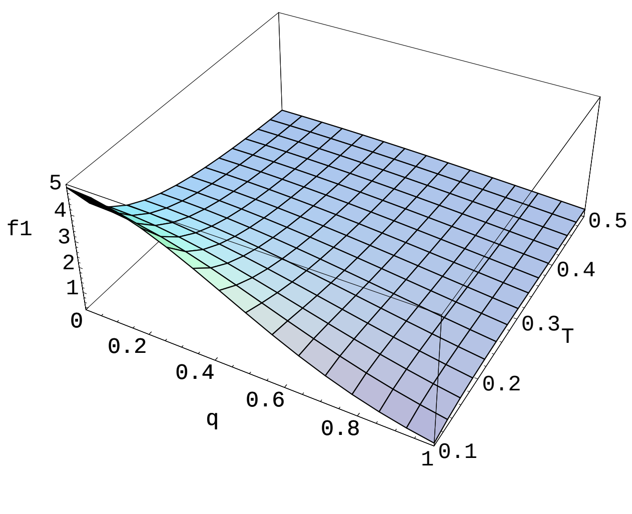

Fig. 1. The pitch angle distribution $f_1(\theta, \tau)$ for $\theta_0 = 0$ in the range of $0 < \theta < 1$, and $0.1 < \tau < 0.5$.

3 The second order pitch angle approximation

In the second order pitch angle approximation one must hold also term of $\mathcal{O}(\theta^2)$. It means that $\sin\theta \to \theta$ and $\cos\theta \to (1-\theta^2/2)$. The equality $\sin^2\theta + \cos^2\theta = 1$ is fulfilled up to the second order of θ, i.e. $\theta^4 \to 0$, and Eq. (2) for f_2 in the second order approximation reads

$$\begin{aligned}&\partial_t f_2 + v\left(1-\frac{\theta^2}{2}\right)\partial_x f_2 \qquad (8)\\ = \ &\frac{v}{\lambda\theta}\,\partial_\theta\,\theta\,\partial_\theta f_2 + \delta(x)\,\delta(t)\,\frac{\delta(\theta-\theta_0)}{\theta_0}\,.\end{aligned}$$

After the Laplace - Fourier transform (in the units of $\{\tau, y\}$) one obtains the equation for $f_2(s, k, \eta, \eta_0)$:

$$\frac{d}{d\eta}\,\eta\,\frac{d}{d\eta}\ f_2 + 2\mathrm{i}k\eta\ f_2 - (s+\mathrm{i}k)\ f_2 = -\frac{1}{2\lambda}\,\delta(\eta-\eta_0)\,, \qquad (9)$$

where $\eta = \theta^2/4$. Solution of Eq. (9) can be expected in the generalized form of the previous solution (6), (7). For this purpose the solution is searched in the form

$$\begin{aligned}f_2 \ = \ &\frac{1}{2\lambda}\int_0^\infty \exp[-\alpha(\eta+\eta_0)\cosh\xi]\left(\coth\frac{\xi}{2}\right)^\beta \qquad (10)\\ \times\ &I_0(2\alpha\sqrt{\eta\eta_0}\sinh\xi)\,\mathrm{d}\xi\end{aligned}$$

with unknown parameters α, β. These are determined by putting function (11) into Eq. (9):

$$\alpha = \sqrt{-2\mathrm{i}k}, \qquad\qquad \beta = -\frac{s+\mathrm{i}k}{\sqrt{-2\mathrm{i}k}}\,. \qquad (11)$$

Conditions for convergence of the integral (11) are the following: $\mathrm{Re}\ \alpha > 0,\ \mathrm{Re}\ \beta < 0$. Then $\alpha = (1-\mathrm{i})\sqrt{k}$ for $k > 0$ and $\alpha = (1+\mathrm{i})\sqrt{k}$ for $k < 0$, respectively. The inverse Laplace transform exists under condition $\mathrm{Re}\ s > 0$, therefore, the condition $\mathrm{Re}\ \beta < 0$ is fulfilled for those α and one obtains

$$\begin{aligned}f_2(k,\tau,\eta,\eta_0) \ = \ &\frac{1}{2\lambda}\ \exp\left[-\mathrm{i}k\tau - \alpha(\eta+\eta_0)\ \coth(\alpha\tau)\right]\\ \times\ &I_0\left(\frac{2\alpha\sqrt{\eta\eta_0}}{\sinh(\alpha\tau)}\right)\frac{2\alpha}{\sinh(\alpha\tau)}\,. \qquad (12)\end{aligned}$$

The function of the argument k in Eq. (12) is regular relative the complex value of k, so, one can apply the inverse Fourier transform of (12). The resulting $f_2(y, \tau, \eta, \eta_0)$ is

$$\begin{aligned}f_2 = \frac{1}{8\pi\lambda}\int_0^\infty &\Big\{\exp\Big[-\mathrm{i}k(y-\tau) - (1+\mathrm{i})\sqrt{k}(\eta+\eta_0)\\ \times\ \coth&\left((1+\mathrm{i})\sqrt{k}\tau\right)\Big]\ I_0\left(\frac{(1+\mathrm{i})\sqrt{k}\sqrt{\eta\eta_0}}{2\sinh\left[(1+\mathrm{i})\sqrt{k}\tau\right]}\right)\\ \times\ &\frac{2(1+\mathrm{i})\sqrt{k}}{\sinh\left[(1+\mathrm{i})\sqrt{k}\tau\right]}\ +\ C.C.\Big\}\,\mathrm{d}k\,, \qquad (13)\end{aligned}$$

where $C.C.$ denotes the complex conjugate term. This expression can be easily rewritten into real arguments, but it is not included here (Shakhov and Stehlik, 2001).

4 The particle density

Let particles are emitted by the point instantaneous source into direction of $\theta_0 = 0$. Then a simple exact expression for the particle density, $\widetilde{N}(x,t)$ or $n(y,\tau)$ can be obtained. Generally, the particle density was defined by the equation

$$\begin{aligned} N(y,\tau) &= \int_{-1}^{1} \mathrm{d}\mu\, f(y,\tau,\mu,\mu_0) \qquad (14) \\ &= \int_0^{\pi} \mathrm{d}\theta\, \sin\theta\, f(y,\tau,\theta,\theta_0)\,. \end{aligned}$$

Here in the small angle approximation one has the Fokker - Planck scattering operator in the form of

$\theta^{-1}\,\partial_\theta\,\theta\,\partial_\theta\,,$

which produces solution quickly decreasing for $\theta \to \infty$, so, the integration limits can be expanded to infinity. Then the density $N(y,\tau;\theta_0 = 0)$ is approximately

$$N(y,\tau;0) = \int_0^{\infty} \mathrm{d}\theta\,\theta\, f_2(y,\tau,\theta,0)\,. \qquad (15)$$

Substituting $f_2(x,t,\theta,0)$ from Eq. (13) in case of zero θ_0 (or η_0) into (15), one obtains the expression for particle density, $N(y,\tau)$,

$$N = \frac{1}{2\pi\lambda}\int_0^{\infty} \mathrm{d}k \left[\frac{\exp\left[-\mathrm{i}k(y-\tau)\right]}{\cosh\left[(1+\mathrm{i})\sqrt{k}\tau\right]} + C.C.\right]. \qquad (16)$$

Numerical calculation shows that density $N(y,\tau)$ tends to zero for all $y = \tau$, i.e. on the position which the particles can achieve without scattering.

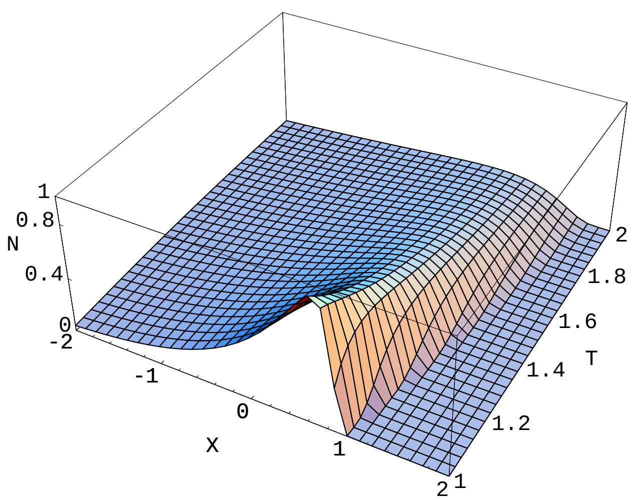

Fig. 2. The particle density $N(y,\tau)$ for $\theta_0 = 0$.

5 Discussion and conclusion

Let one examine the obtained solution in detail. First, the simple solution of the first approximation, (6) or (7), does not correctly involves the scattering together with convection. In fact, all particles freely propagate along x with velocity v, although the pitch angle distribution becomes rather wide. Note here that integral over all angles θ, $\int \mathrm{d}\theta\,\theta\, f(y,\tau,\theta,\theta_0)$ is equal to $\delta(y-\tau)$ in units of $1/\lambda$ independently on y (or τ) or θ_0.

On Fig. 1 the pitch angle distribution, $f_1(\theta)$ was illustrated at $y = \tau = 0.1 - 0.5$ (in dimensionless units) for $\theta_0 = 0$. The distribution becomes almost flat already at instant $\tau \approx 0.2$ and one very weakly depends on $\theta_0 = 0$.

Unlike the first approximation, the function $f_2(\theta)$ describes the initially anisotropic stream during a larger time past the particle injection, and one has low level at $y \approx \tau$, especially for $y \gg 1$. The space distribution, $f_2(y)$, is demonstrated on Fig. 3, where the pitch angle θ is fixed, $\theta = 0$. The picture is similar for non-zero θ. Particles no more form a group at $y = \tau$, but the space distribution possess rather wide 'tail' behind the front of first particles at $y = \tau$. Its width increase with increasing time, and maximum decreases in amplitude and becomes later with increasing time. Temporal development of $f_2(y,\tau)$ can be seen on Fig.4 for for $\theta = 0.1$, $\theta_0 = 0$ Note that the shape of $f_2(y,\tau)$ very weakly depends on value of θ and, it is equal to zero for $y > \tau$.

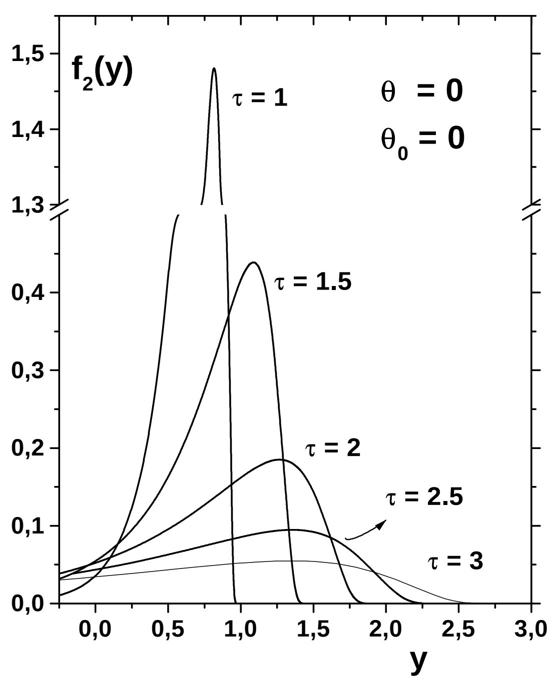

Fig. 3. The space distribution $f_2(y)$ at time $\tau = 1, 1.5, 2, 2.5$ and 3 for $\theta = \theta_0 = 0$.

The particle density (16) is demonstrated on Fig. 2 in the case of particle injection direction of $\theta_0 = 0$.

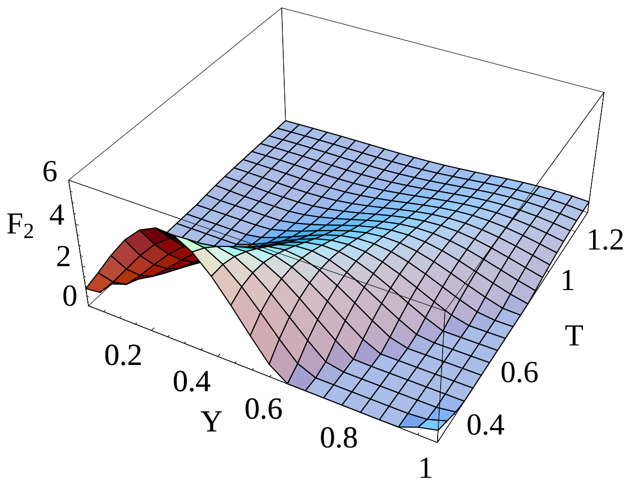

Fig. 4. The space distribution $f_2(y, \tau)$ in the interval $\tau = 0.3 - 1.3$ for $\theta = 0.1$, $\theta_0 = 0$.

We conclude that unlike the first approximation in pitch angle the derived expressions for the particle distribution function in the second approximation as well as the particle density gives more realistic picture of the pitch angle distribution after an unidirectional immediately particle injection.

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